Annexure - A

University Grants Commission

(Ministry of Human Resource Development, Govt. of India)

Bahadurshah Zafar Marg, New Delhi – 110002

Detailed Project completion Report

of

Major Research Project

(MRP-MAJOR- COMP- 2013- 39460)

Entitled, "**Study of Quantum Neural Network (QNN) for Pattern recognition**"

(Sanctioned vide letter no: 43-279/2014(SR))

To

The Principal Investigator

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Detailed Project completion Report

1. **Title of the Project:** Study of Quantum Neural Network (QNN) for Pattern recognition.
2. **Department Involved:** Department of Computer Science
3. **Name of the University Involved:** Dr. B.R. Ambedkar University Agra
4. Mob. No. 9319102434
5. **Date of Implementation:01/07/2015**
6. **Objectives Originally Proposed and Approved:**

In view of the new emerging fields of Quantum Computation [1], Artificial intelligence [2], Quantum Information Processing [3], Entanglement [4] and Distillation of Quantum states [5], Purity and Fidelity of Quantum Channels [6],Quantum Dense Coding [7] and Quantum Cryptography [8], the main objective of the proposed study in this projecthas been to use the orthonormal complete set of our newly discovered Maximally Entangled States (Singh-Rajput MES) [9] for developing the consistent and reliable theory of Quantum Neural Network (QNN) that can give us completely new computational capability for tackling problems which cannot be solved, even in principle, by using classical ArtificialNeural Networks (ANN).It was proposed to work out the correspondence between evolution of these maximally entangled states (Singh-Rajput MES) of two-qubit system and representation of SU(2) group, and to investigate the evolution of MES under a rotating magnetic field. The investigation of the role of entanglement in QNN through these states was also proposed. It was also proposed to develop the appropriate operators, entangled states and quantum associative memory (Qu-AM) for pattern recognition, pattern classification, pattern association, pattern recall, pattern completion and competitive learning in QNN. It was proposed to achieve the amalgamation of intelligence calculation (soft calculation), evolution calculation and quantum calculation and the simulation on QNNs to determine theirarchitecture, function and enormous computational power over their classical counterparts (ANNs).

We also proposed to extend our earlier work [10,11] on hybrid evolutionary neural network to develop a hybrid quantum neural network (HQNN) model based on quantum neurons and traditional neurons where network will include three layers: input layer composed of traditional neurons to receive input information; hidden layer composed of quantum neurons (qubits) to extract pattern feature of input information and transform them to output layer; and output layer composed of traditional neurons to export calculation results. In this model the weightings of output layer will be rectified by back propagation algorithm of Hopfield model and those of hidden layer will be rectified by a group of quantum gates. It was proposed to design a learning algorithm for this model and to illustrate the availability of the model and algorithm by applications in pattern recognition and functional approximation. We also proposed to develop quantum genetic algorithm (QGA) in QNN and to carry out the study of its application in blind source separation (BSS) by using the methods of qubit crossover and qubit rotation strategy. It has been proposed to apply this algorithm to combinational optimization problem of QNN, Pattern Recognition, Functional Approximation, and Pattern Storage and for efficient Pattern Recalling.

It was also proposed to develop QNN with some speculative physical systems like quantum dots. Assembling XOR along with single qubit operations for the system of quantum dots, it was proposed to use these qubits to prepare the maximally entangled states by the method of inclusion, exclusion or phase inversion and to use these maximally entangled states in the theory of QNN. We also proposed to use Majorana fermions to encode quantum information in QNN in a way to solve the problems of dogging quantum computing (destruction of current carriers of quantum bits by small disturbance from local environment). It is also proposed to explore the Implementation of Universal quantum Perceptron model for generalize pattern classification.

7. Achieved Objectives

We have recently explored [12] the entanglement as one of the key resources required for quantum computation and quantum neural networks [QN], established the functional dependence of the entanglement measures on spin correlation functions, worked out the correspondence between evolution of maximally entangled states (MES) of two-qubit system and representation of SU(2) group, and investigated the evolution of MES under a rotating magnetic field. Necessary and sufficient conditions for the general two-qubit state to be maximally entangled state (MES) have been obtained and a new set of MES (Singh-Rajput MES) [9]constituting a very powerful and reliable Eigen basis (Singh-Rajput Eigen Basis) (different from magic bases) [13] of two-qubit systems has been constructed. In terms of the MES constituting this basis, Bell's States have been generated and all the Q-bits of two-qubit system have been obtained. It has been shown that such a MES corresponds to a point in the SO(3) sphere and an evolution of MES corresponds to a trajectory connecting two points on this sphere. Analysing the evolution of MES under a rotating magnetic field, it has been demonstrated that a rotating magnetic field is equivalent to a three dimensional rotation in real space leading to the evolution of a MES. We have also performed [14,15,16] the pattern association (quantum associative memory) and pattern classifications [17, 18,19] by employing the method of Grover's iteration [20] on Bell's MES and **Singh-Rajput MES** in two-qubit system and demonstrated that for all the related processes (memorization, recalling, and pattern classification) in a two-qubit system **Singh-Rajput MES** provide the most suitable choice of memory states and the search states. Using Singh-Rajput MES as memory states in the evolutionary process [20]of pattern storage and the evolutionary as well as non-evolutionary processes of pattern recall (the two fundamental constituents of QuAM) [21], the suitability and superiority of these MES over Bell's MES have been demonstrated [16,17] in both these processes. The whole work carried out in the project period may be classified in the following sections.

A) NECESSARY AND SUFFICIENT CONDITIONS FOR A TWO-QUBIT STATE TO BE MAXIMALLY ENTAINGHLED STATES (MES)

Starting with the theoretical basis of quantum computing, entanglement has been explored as one of the key resources required for quantum computation, the functional dependence of the entanglement measures on spin correlation functions has been established and the role of entanglement in implementation of QNN has been emphasized [12]. It has been shown that the degree of entanglement for a two-qubit state depends on the extent of fractionalization of its density matrix and that the entanglement is completely a quantum phenomenon without any classical analogue. A reliable measure of entanglement of two-qubit states has also been expressed in terms of concurrence [22,23] and it has been shown [12] that in a free two-qubit system the states with both combinations of parallel spins (i.e. states with maximum Hamming spread) are definitely maximally entangled states (MES) while among the states with minimum Hamming spread, those with both anti-parallel combinations are MES and those with one combination of parallel spins and other with anti-parallel spins are not entangled at all.

It has been demonstrated [12] that the general two- qubit state

$|\Psi> = \frac{1}{\sqrt{\gamma}}[\,a|00> \ +b|01> +c|10> \ +d|11>]$ (1)

$$= \frac{1}{\sqrt{\gamma}}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

where $\gamma = |a|^2 + |b|^2 + |c|^2 + |d|^2$,

is maximally entangled when

$$2|ad - bc| = |a|^2 + |b|^2 + |c|^2 + |d|^2$$

or$|(a \mp d^*)|^2 + |(b \pm c^*)|^2 = 0$

Thus we got the following two sets of MES:

$$|\Psi_1> = \frac{1}{\sqrt{(2[|a|^2+|b|^2\}}}[\,a|00> \ +b|01> -b^*|10> \ +a^*|11>] \qquad (2)$$

and $$|\Psi_2> = \frac{1}{\sqrt{(2[|a|^2+|b|^2\}}}[\,a|00> \ +b|01> +b^*|10> \ -a^*|11>] \qquad (3)$$

On Substituting $(a = 1, b = 0); (a = -i, b = 0); (a = 0, b = 1); and\ (a = 0, b = -i)$, the following Bell states (i.e. magic bases) may readily be obtained from the state $|\Psi_1>$ of equation (2):

$|\phi_1> = -\frac{i}{\sqrt{2}}(\,|00> -|11>);$ $\quad|\phi_2> = \frac{1}{\sqrt{2}}(\,|00> +|11>)$

$|\phi_3>=-\frac{i}{\sqrt{2}}(\ |01>+|10>\ )$; $|\emptyset_4>=\frac{1}{\sqrt{2}}(\ |01>-|10>\ ),$ (4)

which are well known maximally entangled orthonormal states constituting magic eigen basis [13].Other maximally entangled two-qubit states which form the orthonormal complete set (i.e. eigen basis) may be obtained as follows by putting $a=\pm 1\ and\ b=1$ in state $|\Psi_2>$ of equation (2.16) and $a=1, b=\pm 1$ in state $|\Psi_1>$ of equation (2);

$$|\psi_1>=\frac{1}{2}[-|00>\ +|01>+|10>\ +|11>],$$

$|\psi_2>=\frac{1}{2}[\ |00>\ -|01>+|10>\ +|11>],$

$|\psi_3>=\frac{1}{2}[\ |00>\ +|01>-|10>\ +|11>],$

$|\psi_4>=\frac{1}{2}[\ |00>\ +|01>+|10>\ -|11>]$ (5)

The concurrence for each of these states is unity and these states constitute the orthonormal complete set since

$<\psi_\mu|\psi_\nu>=\delta_{\mu\nu}$

and$\sum_{\mu=1}^{4}|\psi_\nu><\psi_\mu|=I$

Thus the set of Bell states is not the only eigen basis (magic eigen basis) of the space of two- qubit system but the set of MES given by eqns. (5) also constitute a very powerful and reliable eigen basis of two-qubit systems. This is the new eigen basis and to differentiate it from the already known Bell's basis we have designated it as *Singh-Rajput basis* for its possible use in future in the literature. The MES constructed in the form given by eqns. (5) have been correspondingly labelled as *Singh-Rajput states*. In this new eigen basis, various qubits of two-qubit states have been written as:

$|00>=\frac{1}{2}\ [\ |\psi_2>\ +|\psi_3>\ +\ |\psi_4>\ -|\psi_1>],$

$|01>=\frac{1}{2}\ [\ |\psi_1>\ +|\psi_3>\ +\ |\psi_4>\ -|\psi_2>],$

$|10>=\frac{1}{2}\ [\psi_1>\ +|\psi_2>\ +\ |\psi_4>\ -|\psi_3>],$ (6)

$|11>=\frac{1}{2}\ [\ |\psi_1>\ +|\psi_2>\ +|\psi_3>\ -\ |\psi_4>]$

and accordingly the Bell states have beenconstructed as follows in this new basis;

$|\phi_1>=\frac{-i}{\sqrt{2}}[\ |\psi_4>-|\psi_1>];\ |\phi_2>=\frac{1}{\sqrt{2}}[\ |\psi_2>\ +|\psi_3>];$

$|\phi_3>=\frac{-i}{\sqrt{2}}[|\psi_4>\ +|\psi_1>];\ \ |\phi_2>=\frac{1}{\sqrt{2}}[\ |\psi_3>\ -\ |\psi_2>]$ (7)

It has been demonstrated that any of these new maximally entangled states (Singh-Rajput MES) corresponds to a point in the SO(3) sphere, an evolution of MES corresponds to a trajectory connecting two points, and the initial state $|(1,0)>_+$ locates the centre of the SO(3) sphere.Thus entanglement has been explored as one of the key resources required for quantum computation, the functional dependence of the entanglement measures on spin correlation functions has been established, and correspondence between evolution of MES of two-qubit system and representation of SU(2) group has been worked out in the new eigen basis (*Singh-Rajput eigen basis*). Analysing the evolution of MES under a rotating magnetic field, it has been demonstrated that a rotating magnetic field is equivalent to a three dimensional rotation in real space leading to the evolution of a MES.Analysing the role of entanglement in implementation of quantum neural networks (QNN)the correct computation of XOR function has been carried [12]out in neural network and it has been shown that QNN requires the proper correlation between the input and output qubits and the presence of appropriate entanglement in the system guarantees this correlation.It has been emphasized that the newly constructed maximally entangled two-qubit states (Singh-Rajput MES), constituting new eigen basis, may be the most appropriate choice for utilizing entanglement in quantum neural computation. It has been shown that in quantum approach to neural networks all patterns can be stored as a superposition, where each of the patterns can be considered as existing in a separate quantum universe .It has also been shown that in neural networks the integrity of a stored pattern (bases states) is due to entanglement and the quantum associate memory (Qu AM) is the realization of the extreme condition of many Hopfield networks each storing a single pattern in parallel quantum universes.

B) **PATTERN CLASSIFICATION IN TWO-QUBIT AND THREE- QUBIT SYSYEMS**

Pattern classifications have been performed [17,18] in the straight forward approach employing the method of Grover's iterate [24] on Bell's MES [25] and Singh-Rajput MES in two-qubit system. It has been demonstrated that none of the maximally entangled Bell's state is suitable for correct pattern classification of the point 0? (where ? stands for 0 or 1) upon measurement of two-qubit system on various iterations of Grover's search while the first two maximally entangled states of Singh-Rajput basis, given by eqns. (5), are the most suitable choice for the desired pattern classification. It has also been demonstrated that any of the other two maximally entangled states of Singh-Rajput basis (third and fourth MES) is the most suitable choice as search state for the desired pattern classification '1?' based on Grover's iterative search algorithm in two-qubit system while the probability of correct desired pattern classification in this case also never exceeds the limit of fifty percent when any of the Bell's maximally entangled state is chosen as search state. Performing pattern classifications of points '00' and '11' respectively based on Grover's iterative search algorithm, it has been demonstrated that for any pattern classification in a two-qubit system the maximally entangled states of Singh-Rajput Eigen basis provide the most suitable choice of search states and in no case any of Bell's states is suitable for such classifications.

After constructing the maximally entangled states, the pattern classification has been performed in straight forward approach employing the method of Grover's iterate which is described as a product of unitary operators $\hat{G}\hat{R}$ applied to quantum state iteratively and probability of desired result maximized by measuring the system after appropriate number of iterations. Here the operator $\hat{R}$ is phase inversion of the state(s) that we wish to observe upon measuring the system. It is represented by identity matrix I with diagonal elements corresponding to desired state(s) equal to -1 and the operator $\hat{G}$ descried as an inversion about average:

If $|\Psi> = \frac{1}{\sqrt{2^{n+1}}} \sum_{(x_i y_i) \in B^n} |x_i\, y_i>$ then G=2$|\Psi><\Psi| - I$

For a Two qubit system we have

$|\Psi> = \frac{1}{4}[|00> + |01> + |10> + |11>]$,

$$\hat{G} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \qquad (8)$$

Wehave found the probability of observing the correct classification of the point '0?', where ? denotes 0 or 1, upon measurement on each iteration of Grover's search applied to given state.

For the given search point the involved qubits are $|00>$ and $|01>$ and therefore the phase inversion operator $\hat{R}$ is given by

$$\hat{R} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus we have the following iteration operator

$$\hat{D} = \hat{G}\hat{R} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{bmatrix} \qquad (9)$$

The first iteration of state $|\psi_1>$ of eqns. (5) by operator $\hat{D}$ leads to the pure state $|01>$ with the following probabilities of desired classification;

$$\boldsymbol{P_C = 0;\ P_W = 1;\ P_R = 0; P = 1\ and\ P_{cond.} = 0}$$

where P_C and P_W are the probabilities of correct classification (00) and incorrect classification (01), respectively;P is total probability of the classification '0?',

$P = P_C + P_W$;

P_R is the probability of irrelevant classification (other than that in which we are interested); andP_{cond} is the conditional probability (if the desired pattern is classified then the probability that the classification will be the correct one)

$$P_{cond} = \frac{P_C}{P_C + P_W}$$

Second iteration of state $|\psi_1>$ by operator $\widehat{D}$ leads to the second MES $|\psi_2>$, with reversed sign, of Singh-Rajput basis. It gives the following probabilities of desired classification;

$$P_C = 0.25;\ P_W = 0.25;\ P_R = 0.5; P = 0.5\ and\ P_{cond.} = 0.5$$

Third iteration leads to the pure state -$|00>$ with the following probabilities of desired classification;

$$P_C = 1;\ P_W = 0;\ P_R = 0; P = 1\ and\ P_{cond.} = 1$$

Fourth iteration restores the state $|\psi_1>$ and the same periodicity is repeated in further iterations. *Thus on the third iteration the first state of Singh-Rajput basis gives the 100% probability of the correct pattern classification with the 0% probability of irrelevant classification (other than that in which we are interested).Though the probability of correct classification on first iteration is zero but the total probability of classification of the desired pattern '0?',* $P = P_C + P_W$*,is hundred percent and the* probability *of irrelevant classification is zero percent*.

The similar periodicity of probability of desired pattern is observed by choosing the second maximally entangled Singh-Rajput state $|\psi_2>$ with the hundred percent probability of correct pattern classification after the first iteration and hundred percent total probability of classification of the desired pattern and zero percent probability of irrelevant classification after the first and third iterations.

It shows that these states $|\psi_1>$ *and* $|\psi_2>$ *of Singh-Rajput basis are most suitable choice for desired pattern classification*$|0?>$ *for a two-qubit system.*

On the other hand, if $|\psi_3>$ and $|\psi_4>$ of eqns. (5) are chosen as the search states then also the behaviour of probabilities after third iterations is repeated periodically but the probability of correct classification never exceeds beyond 25% and the total probability of desired classification does not exceed beyond 50% while the probability of irrelevant classifications never falls below 50%. Thus these states are not good choice as search state for the desired pattern classification '0?' in two-qubit system. The comparative periodic behaviours of probabilityP_C of correct pattern classification on different iterations of all the maximally entangled states of equation (5) are shown in figure-1 where red line is for $|\psi_2>$, blue for $|\psi_1>$ and green and violet for $|\psi_3>$, $|\psi_4>$ respectively.

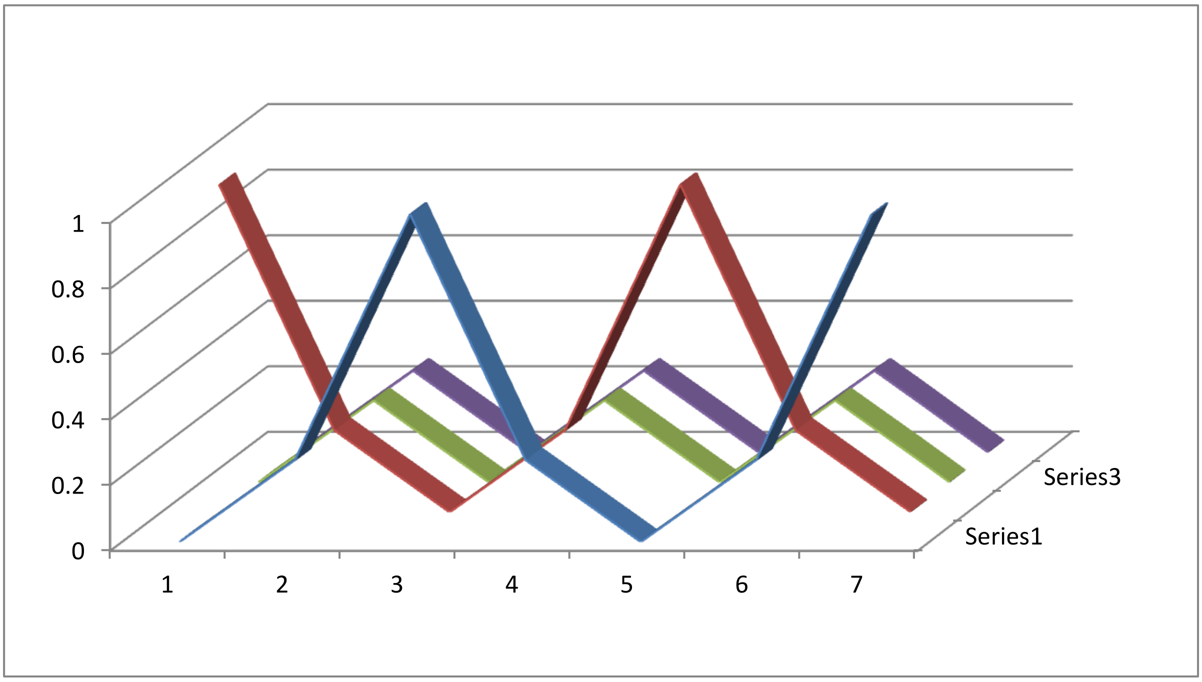

Figure-1: Probability of Correct Classification '0?'for Singh-Rajput States

On the other hand, it has been shown that the probability of correct pattern classification based on any of Bell's state does not exceed the limit of fifty percent on any number of iterations and *hence these states (Bell's states)are not the suitable choice as search state for the pattern classification based on the Grover's iterative search algorithm for a two-qubit system.*

We have also classify the pattern '00' i.e. the pure state, based on Grover's iterative search algorithm by using Singh-Rajput MES. For this classification the following inversion operator and the iteration operators have been obtained as

$$\hat{R} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\hat{D} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \qquad (10)$$

respectively. It has been demonstrated that state $|\psi_1>$ of eqns.(5) gives hundred percent probability of correct classification on second and fifth iterations by the operator of eqn. (10) and *hence the choice of the first state $|\psi_1>$ of Singh-Rajput basis as the search state is most suitable for the desired classification of the pattern* $|00>$. On the other hand the probability of the correct desired pattern classification is not better than 25% with any other state of Singh-Rajput basis. In our attempt to classify the point '00' i.e. the pure state, based on Grover's iterative search algorithm by using any of the Bell state constituting the so called magic basis, we find that the probability of correct pattern classification does not exceed beyond fifty percent on any number of iterations and *hence Bell states are not the suitable choice for the classification of this pattern also.*

We have also classified the pattern ‘1?’ (where ? is 0 or 1) based on Grover’s iterative search algorithm by using our new maximally entangled states. The inversion operator and the iteration operator for this case have been respectively obtained as follows

$$\hat{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$\hat{D} = \frac{1}{2}\begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

The first iteration of this operator on the maximally entangled state $|\psi_3>$ of eqn. (5) has been shown to give hundred percent probability of correct pattern classification ‘11’ and the zero percent probability of irrelevant classification while the third iteration gives the hundred percent total probability of the desired pattern classification. On the fourth iteration the state $|\psi_3>$is restored and the same periodic behaviour is repeated in the further iterations.

Choosing the fourth maximally entangled state $|\psi_4>$ of eqns. (5) as the search state,it has been shown that the first iteration by operator $\hat{D}$ gives the hundred percent total probability of the desired classification and the third iteration gives the hundred percent probability of correct pattern classification ‘11’ and the zero percent probability of irrelevant classification.

Thus these two maximally entangled states $|\psi_3>$and $|\psi_4>$ of Singh-Rajput MES are the most suitable choice as search state for the desired pattern classification ‘1?’ based on Grover’s iterative search algorithm in two-qubit system. It has also been shown that the probability of correct pattern classification never exceeds the limit of fifty percent when any of the Bell’s maximally entangled state is chosen as search state. It has also been shown that the most suitable choice for the desired pattern classification ‘11’, based on Grover’s iterative search algorithm, is the fourth maximally entangled state $|\psi_4>$ of Singh-Rajput eigen basis for a two-qubit system.

In an attempt to make the practical applications of our results of pattern classifications in two-qubit system based on our new maximally entangled states, we have very recently undertaken [26] the study of the classification of Apples and Oranges in a warehouse in the framework of quantum neural network (QNN) in a two- qubit system using the method of repeated iterations in Grover’s algorithm [20] and the algorithm of Ventura [27] and taking different superposition of two- pattern start state containing Orange and Apple both, one- pattern start state containing Apple as search state and another one- pattern start state containing Orange as search state.Basic idea of Grover' algorithm is to invert the phase of the desired basis state and then to invert all the basis states about the average amplitude of all the states. The number (r) of times the classification will have to be repeated in Grover's method in a 2- qubit system is r= $\frac{\pi}{4}\sqrt{N} = \frac{\pi}{2}$, where N =4. It gives $1 \le r < 2$.

On the other hand Ventura algorithm may be written for the present case in the following simplified manner;

$$|\Psi > = \hat{G} I_\rho \hat{G} I_\tau |\psi >$$

Repeat $\frac{\pi}{4}\sqrt{N} - 2 \approx 1\ time$

and take $|\hat{\Psi} > = \hat{G} I_\tau |\Psi >$

for measuring the probability of desired classifications where $|\psi >$ is the search state (stored data base),I_τ inverts the sign of the pattern to be classified, I_ρ inverts the signs of all patterns in the stored data base and the operator $\hat{G}$ is given by eqn. (8).Applying Grover's algorithm and Ventura' method on all the possible superposition as the search states obtained for one-pattern start states consisting of the patterns corresponding to Apples and Orangs respectively, It has been shown that the superposition of phase-invariance are the best choice as the respective search state in both Grover's and Ventura's methods of classifications of patterns. These states respectively are identical to the third and fourth states $|\psi_3 >$ and $|\psi_4 >$ of Singh- Rajput MES (maximally entangled states) Thus it has been demonstrated that any of the maximally entangled states $|\psi_3 >$ and $|\psi_4 >$ of Singh-Rajput basis is the most suitable choice as search state for the desired pattern classification '1?' (Apples and Oranges) based on both of Grover's iterative search algorithm and Ventura's repeated search algorithm in two-qubit system.

Based on our foregoing analysis of the pattern classifications in a two qubit system using Grover's iterative search algorithm and Ventura's repeated search algorithm separately, it has been concluded that:

i) First two states $|\psi_1 >$ and $|\psi_2 >$ of our new MES (Singh-Rajput MES) are the most suitable choice as search state for the correct classification of the patterns $|0? >$ where the symbol '?' denotes 0 or 1.

ii) Last two states $|\psi_3 >$ and $|\psi_4 >$ of our new MES are the best choice as search state for the correct classification of the patterns $|1? >$

iii) Our first MES $|\psi_1 >$ is the best choice as the search state for the correct classification of the pattern $|00 >$and our second MES $|\psi_2 >$is the most suitable choice of the search state for the correct classification of the pattern$|01 >$

iv) Our fourth MES $|\psi_4>$is the best choice as search state for the correct classification of the pattern $|11>$ while our third MES $|\psi_3>$ is the most suitable choice of the search state for the classification of the pattern $|10>$

Applying the method of Grover's iterate on three different superposition in three-qubit system, it has been shown [17] that the choice of exclusive superposition, as the search state, is most suitable one for the desired pattern classifications based on Grover's iterative search algorithm. Starting with the set

$T = \{(x_i, y_i)\} = [(000), (111)]$

the superposition of inclusion, exclusion and phase invariance for a three-qubit system have been respectively constructed as

$$|\Psi_{inc}> = \frac{1}{\sqrt{2}}[|000> + |111>$$

$$|\Psi_{exc}> = \frac{1}{\sqrt{6}}[|001> + |010> + |011> + |100> + |101> + |110>]$$

and

$$|\Psi_{ph}> = \frac{1}{\sqrt{8}}[-|000> + |001> + |010> + |011> + |100> + |101> + |110> - |111>] \quad (11)$$

For the classification of the pattern $|00?>$, where the symbol '?' stands for 0 or1,the operator of phase inversion of the states that we wish to observe has been constructed as

$$\hat{R} = \begin{bmatrix} -I_2 & 0_1 \\ 0_2 & I_6 \end{bmatrix}, \quad (12)$$

whereI_2 is unit matrix of order 2×2, I_6 is unit matrix of order 6×6, $0_1$is null matrix of order 2×6 and $0_2$is the null matrix of order 6×2. The operator$\hat{G}$, describing an inversion about average, has been obtained as:

$$\hat{G} = \frac{1}{4}[g_{ij}] , \quad (13)$$

where $g_{ii} = -3$ and $g_{ij} = 1$ for $i \neq j$, with i,j=1,2-----------8

Then the operator for Grover's iterations has been derived as

$$\hat{D} = \hat{G}\hat{R} = \frac{1}{4}\begin{bmatrix} 3 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -3 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -3 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -3 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -3 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -3 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & -3 \end{bmatrix} \quad (14)$$

Iteratively applying this operator on all superposition given by eqns. (11) the probabilities of correct classification of pattern $|000>$, desired classification of pattern $|00?>$ and irrelevant classification (other than that in which we are interested) in all cases have been calculated and the comparative probabilities have been plotted as the graphs of probability versus number of iterations in the following form:

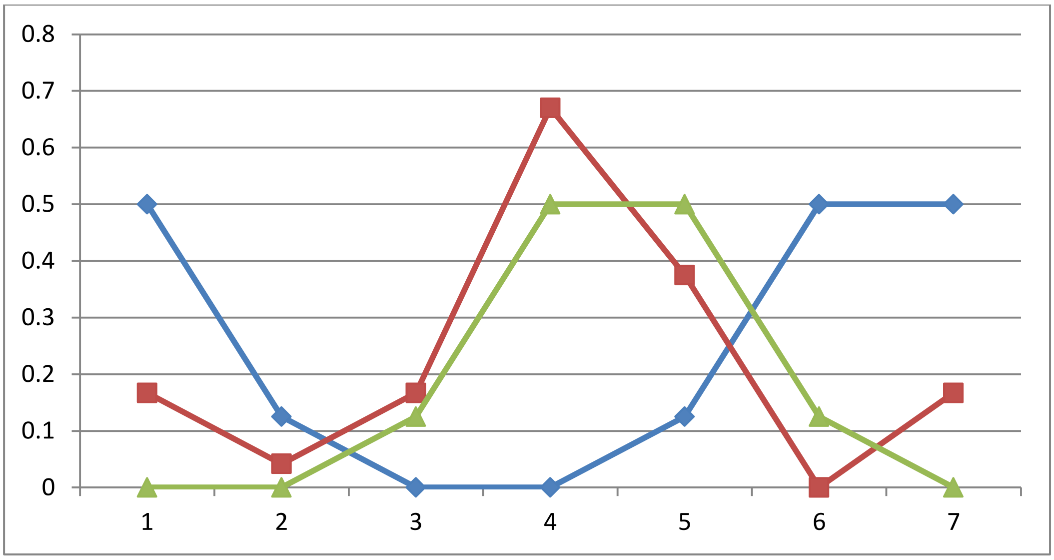

Fig. 2: Probability of Correct Classification '000' for Three Different Superpositionwhere blue line is for the state $|\Psi_{inc}>$, red line for $\Psi_{exc}>$ and green line for $|\Psi_{ph}>$,

which shows the superiority of the search state $|\Psi_{exc}>$ for the desired pattern classification,

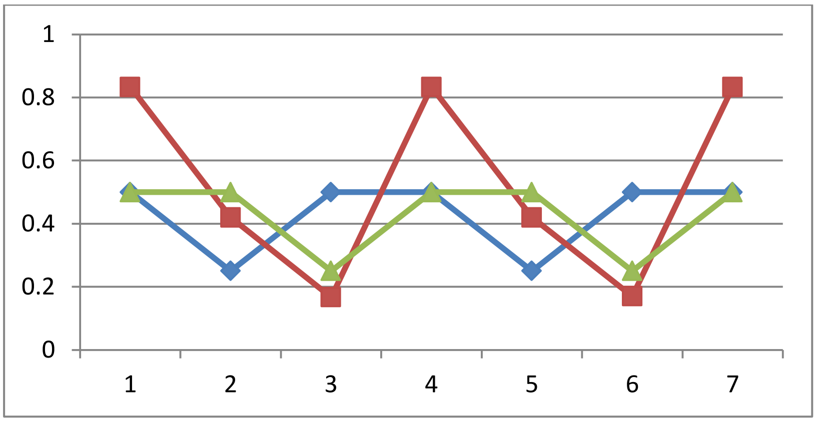

Figure-3: Total Probability of Desired Classification '00?' for Three Different Superposition,
This shows the highest total probability (83%) of desired pattern classification after first and fourth iterations of the superposition $|\Psi_{exc}>$,

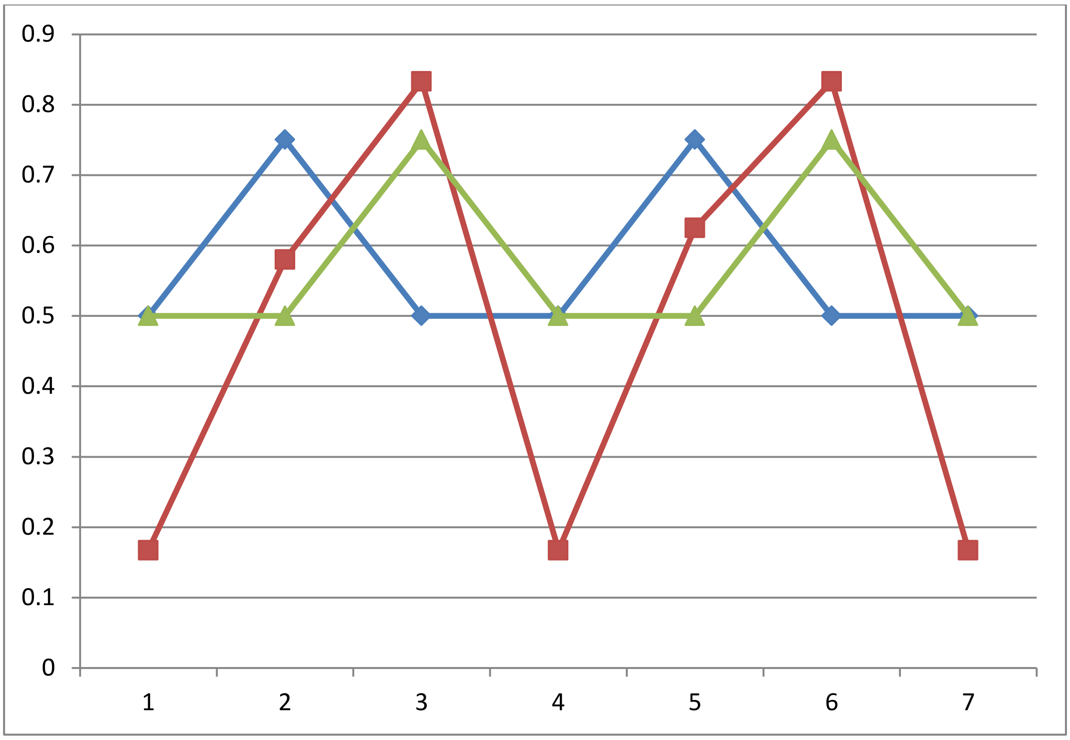

Figure-4: Probability of Irrelevant Classification for Three Different Superposition

This is showing that the probability of irrelevant classification falls to the lowest value of 16.7% on first and fourth iterations of $|\Psi_{exc}>$. All these graphs shows that if the measurement is made after four iterations the choice of the superposition $|\Psi_{exc}>$ as the search state is most suitable for the desired pattern classification based on Grover's iterative search algorithm.

In our attempt of pattern classification of the point '11?', where the symbol '?' denotes 0 or1 with the correct classification as '111', we obtained the phase invariance operator and the iteration operator for Grover's algorithm as

$$\hat{R}=\begin{bmatrix} I_6 & 0_2 \\ 0_1 & -I_2 \end{bmatrix}$$

and

$$\hat{D}=\frac{1}{4}\begin{bmatrix} 3 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -3 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -3 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -3 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -3 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -3 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 3 \end{bmatrix} \qquad (15)$$

It has been interesting to note that results of different number of iterations of this operator on different superposition given by eqns. (11) are the same as those shown in figure-2 to figure-4 for the search of pattern '00?' and in this case also if the measurement is made after four iterations the choice of the superposition $|\Psi_{exc}>$ as the search state is most suitable for the desired pattern classification based on Grover's iterative search algorithm for three-qubit systems.

In our attempt to demonstrate the practical applications of our results of pattern classifications for three-qubit systems, the study of the classification of Apples and Oranges in a warehouse has been undertaken [19] using Grover's method [24] of repeated iterations and Ventura's algorithm [27] separately. In this warehouse the dealer wants a machine with a set of sensors, which measures three properties (parameters) of the fruit: shape, texture and weight:

$$P=\begin{bmatrix} shape \\ texture \\ weight \end{bmatrix}$$

The sensor with output as the shape will give 1 if the fruit is round and 0 if it is elliptical. The texture sensor will give the output 1 if it is smooth and 0 if it is rough and the weight sensor will give the output 1 if weight of the fruit is greater than 1 pound and 0 if weight is less than 1 pound. Therefore, a prototype Orange would be represented by the pattern

$P_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or |100>

and a prototype Apple would be represented by the pattern

$P_2=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ or |110>.

Based on minimum Hamming distance the patterns |000>, |001>, |100> and |101> belong to the class C_1 containing |100> (Orange) and other patterns of the usual three qubit system i.e. |010>, |011>,|110> and |111> belong to the class C_2 containing |110>(Apple). We have found the respective probabilities of classifications of Apples (pattern P_2) and Oranges (pattern P_1) separately by using Grover's algorithm and Ventura's respectively.

Operator describing an inversion about average has been constructed as the following square matrix of order eight

$$\hat{G}=\frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \end{bmatrix} \qquad (15a)$$

The phase inversion operators and corresponding iteration operators for patterns separately representing Apples and Oranges have been derived and various possible superposition as the choice for search states for the classification of these patterns have been obtained for starting states consisting of two patterns and a single pattern respectively. For the classification of pattern P_2, representing Apple, the phase inversion operator $\hat{R}$ has been obtained in terms of the following matrix

$$\hat{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (15b)$$

Using relations (15a) and (15b), we get the following iteration operator of the method of Grover's repeated iterations for classification of the state |110> (Apple):

$$\hat{D} = \hat{G}\hat{R} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -3 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -3 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -3 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -3 \end{bmatrix} \qquad (15c)$$

For the starting states consisting of patterns P_1 and P_2 , the three possible superposition have been constructed as

$$|\Psi inc> = \frac{1}{\sqrt{2}}[|100> + |110>]$$

$$|\Psi exc> = \frac{1}{\sqrt{6}}[|000> + |001> + |010> + |011> + |101> + |111>]$$

and

$$|\Psi phi> = \frac{1}{\sqrt{8}}[|000> + |001> + |010> + |011> - |100> + |101> - |110> + |111>]$$

Applying the iteration operator, given by eqn. (15.c), separately on these superposition repeatedly, the probabilities of correct classification of the pattern $P_2(Apples)$, incorrect classification of other patterns in class $C_2$of pattern P_2 and irrelevant classification (other than that in which we are interested i.e. the patterns not belonging to class C_2) respectively and the total probability of desired classifications and the conditional probability of correct classification of the pattern P_2 have been evaluated and systematically tabulated.The comparative periodic behavior of probabilities of correct classification of pattern P_2 (Apples)for all the three superposition$|\Psi inc>$, $|\Psi exc>$ and $|\Psi \text{phi}>$respectivelyas the search state in the method of Grover's repeated iterations is being shown in the following Fig-5

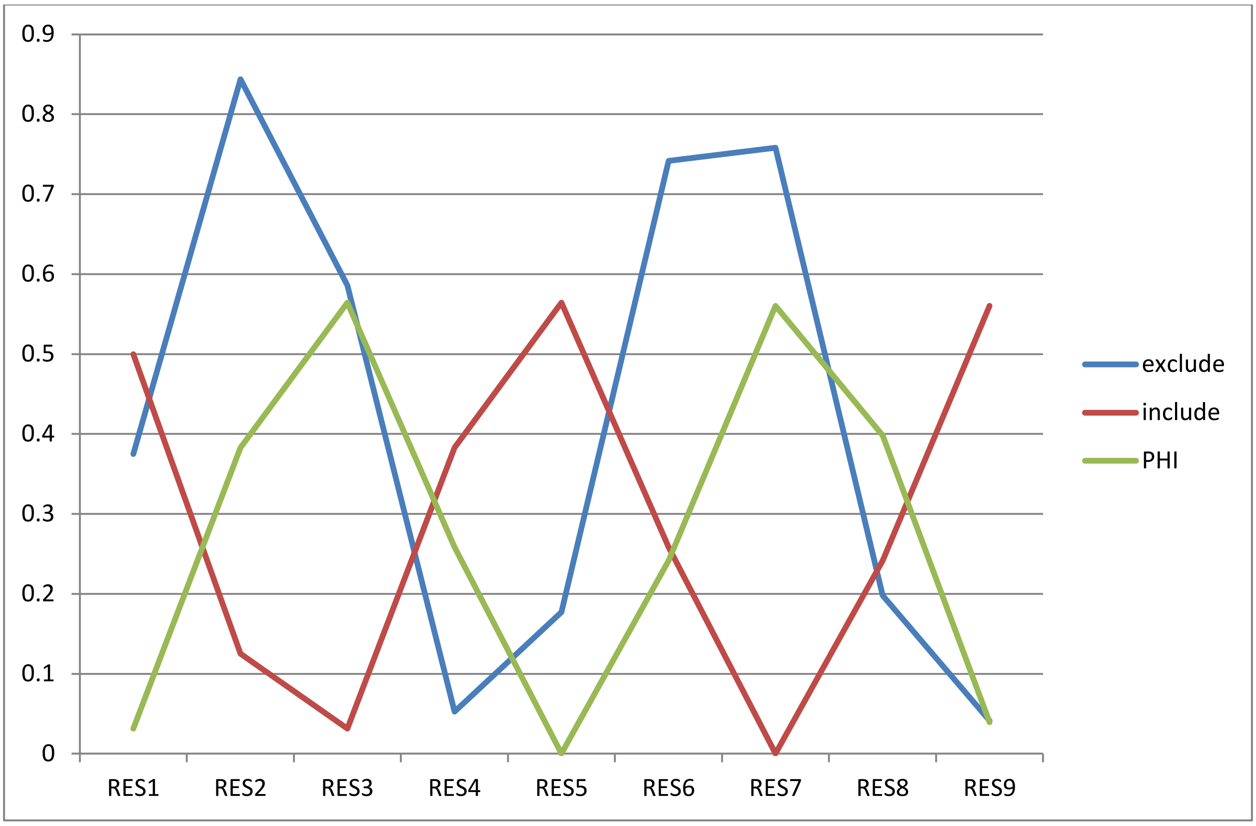

Figure-5: Probability of Correct Classification of 'Apple' for Three Different Superposition

It is observed that the probability of correct classification never exceed the limit of 56% in case of inclusion $|\Psi_{inc}>$and Phase inversion $|\Psi_{phi}>$, whereas in the case of exclusion this probability reaches up to 84% in second iteration. Thus the choice of inclusion superposition $|\Psi_{inc}>$ and phase inversion superposition $|\Psi_{phi}>$as the search state for the desired pattern classification based on Grover's iterative search algorithm are not suitable, whereas *the choice of* $|\Psi_{exc}>$*as search state for the desired pattern classification based on Grover's iterative search algorithm is most suitable if the measurement is made after second iteration.*

The comparative periodic behavior of probability of irrelevant classifications of pattern P_2 for all the three superposition $|\Psi inc>$, $|\Psi exc>$ and $|\Psi$phi >respectivelyas the search state in the method of Grover's repeated iterations is shown in following Fig-6.

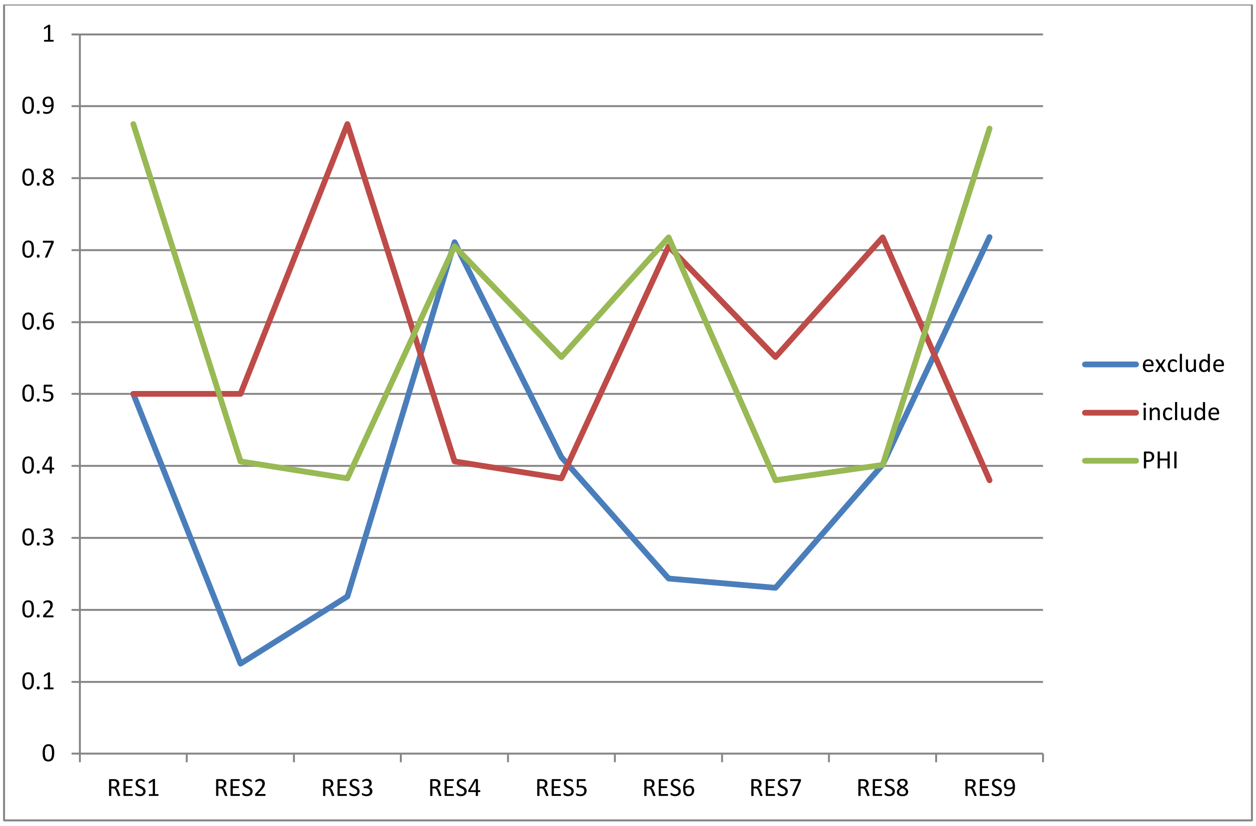

Figure-6: Probability of Irrelevant Classification for Three Different Superposition

It shows that the probability P_R of the irrelevant classifications is as high as 87% on first and ninth iterations if the superposition $|\Psi\text{phi}>$ is chosen as the search state in the method of Grover's repeated iterations for pattern classification. In the case of superposition $|\Psi inc>$ this probability of irrelevant classifications is more than 87.5% on third and eleventh iterations. In view of such high possibilities of irrelevant classifications none of these superposition, $|\Psi\text{phi}>$ or $|\Psi inc>$ is suitable as the search state for the requisite pattern classification. On the other hand, when the superposition $|\Psi exc>$ is chosen as the search state in the process of pattern classification based on Grover's algorithm, the probability of irrelevant classifications never exceeds 71% and it is lowest (negligibly small) on second iteration (when the probability of correct pattern classification is maximum). *Thus the superposition* $|\Psi exc>$ *is the most suitable choice as a proper search state for the correct classification of the pattern* P_2 *on second iteration in the method based on Grover's algorithm.*

All these probabilities of classification of Patten $|110>$(corresponding to Apples) have also been calculated for all the possible superposition with one-pattern start-states consist of the patterns corresponding to Apples and Oranges respectively. Analyzing these results and also their graphical comparative behavior, it has been shown that the superposition of exclusion is the most suitable choice as the search state for classification of Apples by using the Grover's algorithm of repeated iterations. All these probabilities have also been evaluated for the classification of pattern P_1 (Oranges)i.e. $|100>$and it has been shown that on second iteration of the superposition of exclusion by the corresponding iteration operator the pattern P_1 (oranges) is also most suitably classified using the Grover's algorithm with two- patterns and one-pattern start states.

The probabilities of classifications of patternP_2 (Apples), on using Ventura's algorithm [27] for all the possible superposition as the search states, have been calculated and compared with those of Grover's algorithm and it has been demonstrate that in general for classification of a given pattern (Apples) in 3-qubit system the Grover's and Ventura's algorithms are effective in the cases where the number of patterns in the stored data base are larger or smaller respectively. It has been shown that in a 3-qubit system the maximum probability in Ventura's algorithm is not obtained in the case when the number of stored data $= m = \frac{N}{4} + 2 = 4,$ as claimed in an earlier paper [28]. It has also been shown that in Grover's method with any superposition as the search state in a 3-qubit system the probability of classification of the desired pattern for the unknown process (when desired pattern does not belong to the stored data base) is always more than that for the known process (when the pattern to be classified belongs to the stored data base).

Carrying out the classification of pattern in a two-qubit system by separately using Grover's and Ventura's algorithm on different possible superposition it has been shown that the exclusion superposition and the phase-invariance superposition are the most suitable search states obtained from two-pattern start-states and one-pattern start-states, respectively, for the simultaneously classifications of patterns. The higher effectiveness of Grover's algorithm for large search states has been verified but the higher effectiveness of Ventura's algorithm for smaller data base has been contradicted in two-qubit systems and it has been demonstrated that the unknown patterns (not present in the concerned database) are classified more effectively than the known ones (present in the data-base) in both the algorithms. It has also been demonstrated that different states of Singh-Rajput MES obtained from the corresponding self-single-pattern start states are the most suitable search states for the classification of patterns |00>, |01>, |10> and |11> respectively on the second iteration of Grover's method of the first operation of Ventura's algorithm

C) PROCESSES OF QUANTUM ASSOCIATIVE MEMORY (QuAM) THROUGH NEW MAXIMALLY ENTANGLED STATES (SINGH-RAJPUT MES)

Keeping in view that the Quantum Associative Memory (QuAM) is an important tool for pattern recognition, intelligent control and artificial intelligence, we have used our new maximally entangled states(Singh-Rajput MES) [9] as memory states in the evolutionary process of pattern storage [14,16] and the non-evolutionary as well as evolutionary processes of pattern recall [20] (the two fundamental constituents of quantum associative memory) and demonstrated the suitability and superiority of these MES over Bell's MES. It has been shown that, under the operations of all the possible memorization operators for a two-qubit system, the first two states of Singh-Rajput MES are useful for storing the pattern $|11>$ and the last two of these MES are useful in storing the pattern $|10>$ while Bell's MES are not much suitable as memory states in a valid memorization process. The recall operations have also been conducted by separately choosing Singh-Rajput MES and Bell's MES as memory states for possible various queries and it has been shown that in each case the choices of Singh-Rajput MES as valid memory states are much more suitable than those of Bell's MES. The brief account of this whole work, carried out on the processes of quantum associative memory by using our new maximally entangled states, is being given in the following subsections:

i) Process ofPattern Memorization (Storage of Patterns):

The key operator in the Pattern Storage process (memorization operator) is

$$\hat{S}^P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{(\frac{p-1}{p})} & -\frac{1}{\sqrt{p}} \\ 0 & 0 & \frac{1}{\sqrt{p}} & \sqrt{(\frac{p-1}{p})} \end{bmatrix} \qquad (16)$$

where $m \geq p \geq 1$ $with\ m = 4\ for\ a\ two-qubit\ system.$

It is obviously a unitary operator and hence the storage segment of QuAm through this operator is an evolutionary process. This evolutionary nature of storing process is necessary for the system to maintain coherent superposition that represents the stored patterns.

For different values of p this operator has been written as

$$\hat{S}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \hat{S}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{S}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}; \quad \hat{S}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad (17)$$

Each iteration makes use of different $\hat{S}^P$ and results in another pattern being incorporated into the quantum system. Applying the operator $\hat{S}^1$ on our new MES given by eqns. (5), we got

$$\hat{S}^1|\psi_1> = \frac{1}{2}[-|00> + |01> - |10> + |11>] = |\psi_1{}^I>,$$

$$\hat{S}^1|\psi_2> = \frac{1}{2}[|00> - |01> - |10> + |11>] = |\psi_2{}^I>,$$

$$\hat{S}^1|\psi_3> = \frac{1}{2}[|00> + |01> - |10> - |11>] = |\psi_3{}^I>,$$

$$\hat{S}^1|\psi_4> = \frac{1}{2}[|00> + |01> + |10> + |11>] = |\psi_4{}^I>$$

where none of the states $|\psi_1{}^I>, |\psi_2{}^I>$, $|\psi_3{}^I>$ and $|\psi_4{}^I>$ is entangled at all. Thus themaximally entangled nature of Singh-Rajput MES are completely lost after the operation of the operator $\hat{S}^1$ and the magnitude of any pattern is not modified (except the phase change) as per requirement of the recalling process of QuAM in any state.*These results show that the operator $\hat{S}^1$ is not suitable for storing (i.e. memorizing) Singh-Rajput MES as valid memory states.*

With memorising operator $\hat{S}^2$ we got

$$\hat{S}^2|\psi_1> = -0.5\,|00> + 0.5|01> +0.707|11> = |\psi_1{}^{II}>,$$

$$\hat{S}^2|\psi_2> = 0.5\,|00> - 0.5|01> +0.707|11> = |\psi_2{}^{II}>,$$

$$\hat{S}^2|\psi_3> = 0.5\,|00> + 0.5|01> -0.707|10> = |\psi_3{}^{II}>,$$

$$\hat{S}^2|\psi_4> = 0.5\,|00> + 0.5|01> +0.707|10> = |\psi_4{}^{II}>, \qquad (18)$$

We found that all the resulting states $|\psi_1{}^{II}>, |\psi_2{}^{II}>, |\psi_3{}^{II}>$, and $|\psi_4{}^{II}>$ are entangled (though not maximally entangled). Here we observed that the coefficient of pattern $|11>$ is increased and that of pattern $|10>$ vanishes by the operations of the memorization operator $\hat{S}^2$ on the first and second states, $|\psi_1>$ and $|\psi_2>$, while its operation on the last two states$|\psi_3>$ and $|\psi_4>$ enhances the coefficient of pattern $|10>$ and makes the coefficient of the pattern$|11>$ vanishing. *Thus with the memorization operator $\hat{S}^2$ the choice of states $|\psi_1>$ and $|\psi_2>$ of Singh-Rajput MES as memory states may be useful in storing the pattern $|11>$ in QuAM while that of states $|\psi_3>$ and $|\psi_4>$ may be found useful in storing the pattern $|10>$.*

Applying the memorising operator $\hat{S}^3$ on our new MES we got

$$\hat{S}^3|\psi_1> = -0.5\,|00> + 0.5|01> +0.119\,|10> +0.697|11> = |\psi_1{}^{III}>$$

$$\hat{S}^3|\psi_2> = 0.5\,|00> - 0.5|01> +0.119\,|10> +0.697|11> = |\psi_2{}^{III}>,$$

$$\hat{S}^3|\psi_3> = 0.5\,|00> + 0.5|01> -0.697\,|10> +0.119|11> = |\psi_3{}^{III}>,$$

$$\hat{S}^3|\psi_4> = 0.5\,|00> + 0.5|01> +0.697\,|10> -0.119|11> = |\psi_4{}^{III}>, \qquad (19)$$

We found that all the resulting states $|\psi_1{}^{III}>, |\psi_2{}^{III}>, |\psi_3{}^{III}>$ and $|\psi_4{}^{III}>$ are maximally entangled with enhanced coefficient of pattern $|11>$ in memorised states $|\psi_1{}^{III}>$ *and* $|\psi_2{}^{III}>$ and the enhanced coefficient of pattern $|10>$ in the memorised states $|\psi_3{}^{III}>$ and $|\psi_4{}^{III}>$. *Thus with the memorization operator $\hat{S}^3$ also the first two states of Singh-Rajput MES may be useful for storing the pattern $|11>$ while the the last two states of these MES are useful for storing the pattern$|10>$.*

With memorising operator $\hat{S}^4$ we got

$$\hat{S}^4|\psi_1> = -0.5\,|00> + 0.5|01> +0.183\,|10> +0.683|11> = |\psi_1{}^{IV}>,$$

$\hat{S}^4|\psi_2>= 0.5\,|00> - 0.5|01> +0.183\,|10> +0.683|11> \qquad = |\psi_2{}^{IV}>$,

$\hat{S}^4|\psi_3>= 0.5\,|00> + 0.5|01> -0.683|10> +0.183|11> = |\psi_3{}^{IV}>$

$\hat{S}^4|\psi_4>= 0.5\,|00> + 0.5|01> +0.683|10> -0.183|11> = |\psi_4{}^{IV}> \qquad (20)$

We found that the resulting memorized states $|\psi_1{}^{IV}>, |\psi_2{}^{IV}>, |\psi_3{}^{IV}>$ and $|\psi_4{}^{IV}>$ are all partially entangled with the enhanced value of coefficient of pattern $|11>$ in the first two memorized states and the enhanced value of coefficient $|10>$ in the last two memorized states. *Thus with the memorization operator $\hat{S}^4$ also the first two states of Singh-Rajput MES are useful for storing the pattern $|11>$ and the last two states of these MES are suitable for storing the pattern$|10>$.*

On applying the memorization operators of eqns. (17) on Bell's MES, given by eqns. (4), we got

$$\hat{S}^1|\phi_1> = |\phi_3> = |\phi_1{}^I>; \quad \hat{S}^1|\phi_2> = |\phi_4> = |\phi_2{}^I>$$

$$\hat{S}^1|\phi_3> = \frac{-i}{\sqrt{2}}[|01> +|11>] = |\phi_3{}^I>;$$

$$\hat{S}^1|\phi_4> = \frac{1}{\sqrt{2}}[|01> -|11>] = |\phi_4{}^I> \qquad (21)$$

$$\hat{S}^2|\phi_1> = -0.707i|00> -0.5i|10> +0.5i|11> = |\phi_1{}^{II}>;$$

$$\hat{S}^2|\phi_2> = 0.707|00> -0.5|10> +0.5|11> = |\phi_2{}^{II}>;$$

$$\hat{S}^2|\phi_3> = -0.707i|01> -0.5i|10> -0.5i|11> = |\phi_3{}^{II}>;$$

$$\hat{S}^2|\phi_4> = 0.707|01> -0.5|10> -0.5|11> = |\phi_4{}^{II}> \qquad (22)$$

$$\hat{S}^3|\phi_1> = -0.707i|00> -0.41i|10> +0.58i|11> = |\phi_1{}^{III}>;$$

$$\hat{S}^3|\phi_2> = 0.707|00> -0.41|10> +0.58|11> = |\phi_2{}^{III}>;$$

$$\hat{S}^3|\phi_3> = -0.707i|01> -0.577i|10> -0.41i|11> = |\phi_3{}^{III}>;$$

$$\hat{S}^3|\phi_4> = 0.707|01> -0.577|10> -0.41|11> = |\phi_4{}^{III}>. \qquad (23)$$

$$\hat{S}^4|\phi_1> = -0.707i|00> -0.354i|10> +0.612i|11> = |\phi_1{}^{IV}>;$$

$$\hat{S}^4|\phi_2> = 0.707|00> -0.354|10> +0.612|11> = |\phi_2{}^{IV}>;$$

$$\hat{S}^4|\phi_3> = 0.707i|01> +0.612i|10> +0.354i\,|11> = |\phi_3{}^{IV}>;$$

$$\hat{S}^4|\phi_4> = 0.707|01> +0.612|10> +0.354\,|11> = |\phi_4{}^{IV}> \qquad (24)$$

where first two of eqns. (21) show that the memorization operator $\hat{S}^1$ transforms the first two states of Bell' MES in to third and fourth states respectively without any modification in the coefficient of any of the patterns in the superposition but inducing the patterns which were not present in the original states. The last two of the eqns. (21) show that this memorization operator creates two new states $|\phi_3{}^I>$; and $|\phi_4{}^I>$; none of which is entangled at all and hence the operator $\hat{S}^1$ is not the suitable choice as memorization operator for Bell' MES as memory states. Other sets of eqns. (22-24) show that all the other memorization operators $\hat{S}^2, \hat{S}^3$ and$\hat{S}^4$ of eqns. (17) enhance the coefficients of the pattern $|00>$ when first two of Bell's MES are chosen as memory states while these operators enhance the coefficients of pattern $|01>$ in the last two of Bell's MES but in each of the modified states $\phi_\mu^{II}, \phi_\mu^{III}$and ϕ_μ^{IV} for $\mu = 1,2,3,4$ a new pattern, not present in the initial memory state as any of Bell's MES, is created as the spurious or fictitious memory and hence the *Bell's states are not suitable as memory states for memorization process (storage algorithm) of QuAM.*

Thus carrying out the storage element of QuAM by applying all possible memorizing operators for a two qubit system on Singh-Rajput MES one by one, the corresponding sets of modified memorized states have been obtained and it has been demonstrated that under the operations of all the possible memorization operators for a two-qubit system the first two states of Singh-Rajput MES are useful for storing the pattern $|11>$ and the last two of these MES are useful in storing the pattern $|10>$. It has also been demonstrated that Bell's states are not suitable at all as memory states for memorization process (storage algorithm) of QuAM.

ii) Process of Pattern Recall

In the case of storage algorithm evolutionary processes are a necessity since the system must maintain a coherent superposition that represents the stored patterns. On the other hand requiring the recall mechanism to be evolutionary seems to limit the efficiency with which the recall may be accomplished, since the pattern recall mechanism in the QuAM requires the decoherence and collapse of the wave-function of the system. We have carried out [20] the process of pattern recall by using our new MES [9] in both approaches Evolutionary and Non-Evolutionary separately and our main results of these investigations are being mentioned in the following subsections:

a) Evolutionary Approach of Pattern Recall

We have s realised Grover's search algorithm using Discrete Fourier transform (DFT) with the matrix elements given by[29]

$$F_{ab} = e^{2\pi iab/N}$$

where $N = 4, 0 \leq (a,b) \leq 3$. Thus we have constructed the matrix of DFT as

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \tag{25}$$

with transpose conjugate of this DFT matrix given as

$$F^\dagger = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & -i \end{bmatrix} = TF \tag{26}$$

where the matrix T swapping the rows of the DFT F is given by

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (27)$$

The matrices F and T are obviously unitary matrices and the process related with the operations of these matrices are evolutionary. When the matrix T operates on different memories in a two-qubit system, we have

$T\,|00> = |00>;\ T\,|01> = |11>;\ T\,|10> = |10>; T\,|11> = |01>$ (28)

which may also be written as

$T\,|?0> \ |?0>$; where ?= 0, 1

and $T\,|\alpha 1> = |\alpha' 1>$,

where $\alpha = 0,1\ and\ \alpha' = 0\ if\ \alpha = 1\ and\ \alpha' = 1\ if \alpha = 0$

Applying the Swapping- operator of equation (27) on our new MES, given by eqns. (5), we have:

$$T|\psi_1> = |\psi_1>;\ T|\psi_2> = |\psi_4>;$$

$T|\psi_3> = |\psi_3>;\ T|\psi_4> = |\psi_2>$ (29)

showing that the first and third states, $|\psi_1>$ and $|\psi_3>$ respectively, of Singh-Rajput Basis are self- swapped states while the second state $|\psi_2>$ is relabelled as fourth state $|\psi_4>$ and vice-versa, under swapping operator. Thus the maximally entangled nature and the orthonormal property of our new MESare fully retained under the operation of swapping operator of eqn. (27). *In other words our new MES (Singh-Rajput MES) provide a suitable choice as memory states for the memory recall mechanism of QuAM.*

On the other hand, if the swapping operator of eqn. (27) is operated upon the Bell's MES given by eqns. (4), then we get following new states:

$$|\alpha_1> = T|\phi_1> = -\frac{i}{\sqrt{2}}(|00> -|01>);$$

$$|\alpha_2> = T|\phi_2> = \frac{1}{\sqrt{2}}(|00> +|01>)\ ;$$

$$|\alpha_3> = T|\phi_3> = -\frac{i}{\sqrt{2}}\ (|10> +|11>;$$

$|\alpha_4> = T|\phi_4> = -\frac{1}{\sqrt{2}}\ (|10> -|11>$ (30)

which are no more Bell's states and no more MES. In other words Bell's states lose their MES nature on being operated upon by the swapping operator of eqn. (27). *Thus Bell's states do not provide a suitable choice as memory states for memory recall mechanism of QuAM.*

We have also examined the suitability of a MES as memory state $|\psi_0>$for recalling the memory associated with a given partial pattern by the recall mechanism [29]:

$|\psi> = F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{\emptyset}|\psi_0>$ (31)

where the operator $I_\emptyset$ inverts the phase of the state $|\emptyset>$, operator I_M inverts the phase of any state representing a valid memory (it minimizes the effects of spurious memories that develop during recall process), the operator F is represented by the matrix given by eqn. (25) for a two-qubit system and its inverse F^{-1} is given by

$$F^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \quad (32)$$

In this recall process any patterns in the stored memory, that match the query have their phases inverted. We started with the query '0?', where ? represents unknown that matches either 0 or 1. In other words we have the desired outcome to recall the memory pattern whose first qubit is 0 in a two-qubit system. Choosing the first of our new MES, given by eqns. (5) as the memory state $|\psi_0>$,we wrote the given query as an operator $I_{0?}$:

$$\mathrm{I}_{0?}|\psi_0> = \mathrm{I}_{0?}|\psi_1> = \mathrm{I}_{0?}\,(\frac{1}{2}[-|00> +|01> +|10> +|11>])$$

$$= \frac{1}{2}[\,|00> -|01> +|10> +|11>]$$

$$= |\psi_1{}'> \quad (33)$$

and performed the inversion effected by the operator sequence $-F^{-1}I_{\bar{0}}\,F$ where $I_{\bar{0}}$ inverts the phases of the memories representing the query. Thus we get

$$-F^{-1}I_{\bar{0}}\,F|\psi_1{}'> = \frac{1}{2}\begin{bmatrix} -i \\ 1 \\ i \\ 1 \end{bmatrix} = |\psi_2{}'> \quad (34)$$

where no spurious memory pattern (which is not present in the given memory state) is developed. Continuing with the operator sequence of eqn. (31) the phases of all valid memory states, involved in $|\psi_2{}'>$, are inverted as

$$\mathrm{I_M}|\psi_2{}'> = \frac{1}{2}\begin{bmatrix} i \\ -1 \\ -i \\ -1 \end{bmatrix} = |\psi_3{}'> \quad (35)$$

which gives

$$-F^{-1}I_{\bar{0}}\,F|\psi_3{}'> = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = -|\psi_1> \quad (36)$$

where $|\psi_1>$ is the first of our new MES given by eqns(5). Combining all eqns. (33)- (36), we may write the recall mechanism of eqn. (31) as

$$F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{0?}|\psi_1> = -|\psi_1> \quad (37)$$

which shows that *the first of Singh-Rajput MES as memory state is simply rotated by π under the recall process with query '01'.*

In the similar manner we have

$F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{0?}|\psi_2> = -|\psi_1>$; (38)

$F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{0?}|\psi_3> = |\psi_4>$; (39)

$F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{0?}|\psi_4> = |\psi_3>$ (40)

where eqn. (38) shows that with our second MES, given by (5), as valid memory state in the recall procedure of eqn. (31) with the query '0?' the output is our first MES rotated by π in the similar manner as for the first MES as memory state. In other words the recall mechanism does not make any distinction between first two states of Singh –Rajput MES for the query '0?'. It is the most convenient and expected result for this query and hence any of these two states can be the suitable choice for the memory state in recall mechanism with the given query. Relations (39) and (40) show that our third and fourth MES given by eqns. (5) as the choice of valid memory states are interchanged under the recall mechanism of eqn. (31) with the given query. However these states consist of the common memory patterns (with only sign change of one pattern) and hence no spurious, corrupted or fictitious memory pattern is generated by the given query under the recall procedure when the states of Singh-Rajput MES are used as the memory states. *Thus all these states of Singh-Rajput MES are the suitable memory states for the evolutionary recall procedure with the given query '0?'.*On the otherhand, applying the recall mechanism of eqn. (31) on Bell's MES given by eqns. (4), *we found that among the Bell's MES only second state may be the valid choice as memory state with the given query.*

Making the query '1?', represented by the operator $I_{1?}$ and choosing the first state of Singh-Rajput MES in procedure of recall through this query, we have

$F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{1?}|\psi_1> = |\psi_2>$ (41)

showing the generation of second of Singh-Rajput MES in the recall procedure with the given query $I_{1?}$ when the memory state is the first state of Singh-Rajput MES. The memory patterns in both these states are similar with the change of signs in the first and second elements of the matrices representing these states. No spurious states or the corrupt states are generated in this recall procedure and *hence the first of Singh-Rajput MES can be a suitable and valid memory state in the evolutionary recall process for the given query* $|1?>$.

Similarly, for other states of our new MES we have

$F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{1?}|\psi_2> = |\psi_4>$;

$F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{1?}|\psi_3> = -|\psi_4>$;

$F^{-1}I_{\bar{0}}\,FI_M F^{-1}I_{\bar{0}}\,FI_{1?}|\psi_4> = -|\psi_3>$ (42)

Equations (41) and (42) show that the recall mechanism with the given query '1?' in QuAM with Singh-Rajput MES as valid memory states generate second state $|\psi_2>$ for the first memory state $|\psi_1>$ and gives the state $|\psi_4>$ for the memory state $|\psi_2>$ while the memory states $|\psi_3>$ and

$|\psi_4>$are relabelled as inverted states (rotated by π) $|\psi_4>$ and $|\psi_3>$ respectively. *No spurious or fictitious or corrupted state is generated in theevolutionary recall process with any of Singh-Rajput MES as the choice for memory state In the QuAM model for a two- qubit system*. On the other hand choosing the Bell's MES as the memory state in the recall process for the given query '1?',we found that each recall process generates the state with the pattern different from that of the corresponding memory state. In other words the memory pattern of each output is not contained in the corresponding memory state and hence all the generated states in the recall process with Bell's MES chosen as memory states are spurious and fictitious memory states. *Thus, none of the Bell's MES is suitable choice for the valid memory state in the evolutionary recall process with the given query '1?' in QuAM model.*

Applying the recall procedure with the query as point '11' or the pattern $|11>$ with first of Singh-Rajput MES, given by eqns. (5), as the search state we obtained the following result:

$$F^{-1}I_{\bar{0}}\, FI_M F^{-1} I_{\bar{0}}\, FI_{11}|\psi_1> = |\psi_3> \qquad (43)$$

where the generated state in the recall process is the third of Singh-Rajput MES with the similar memory patterns as contained in the chosen memory state $|\psi_1>$. In other words the recall procedure with the given query projects the first of Singh-Rajput MES as the third state $|\psi_3>$ without affecting the chances of observations of the given query $|11>$ and without generating any spurious or fictitious state. *Thus the first of the Singh-Rajput MES, $|\psi_1>$, is the suitable choice as a valid memory state in the evolutionary recalling process with the given query$|11>$.*

Applying the recall procedure with the given query$|11>$ choosing others of Singh-Rajput MES as memory states one by one, we have

$$F^{-1}I_{\bar{0}}\, FI_M F^{-1} I_{\bar{0}}\, FI_{11}|\psi_2> = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

$$F^{-1}I_{\bar{0}}\, FI_M F^{-1} I_{\bar{0}}\, FI_{11}|\psi_3> = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$$F^{-1}I_{\bar{0}}\, FI_M F^{-1} I_{\bar{0}}\, FI_{11}|\psi_4> = -\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad (44)$$

where all the generated states in recall process are non-entangled and do not constitute the complete orthonormal set. In other words all the memory states lose their maximally entangled character in the process of recall with the given query. Thus all these states are the corrupt and fictitious states and *hence none of these three states $|\psi_2>$, $|\psi_3>$ and $|\psi_4>$ of Singh-Rajput MES can be chosen as the valid memory state in the evolutionary recalling process with the given query $|11>$ in QuAM model.*

Applying recall procedure for the first of Singh-Rajput MES as memory states with the query of point '00' representing the partial pattern $|00>$, we have

$$F^{-1}I_{\bar{0}}\ FI_{M}F^{-1}I_{\bar{0}}\ FI_{00}|\psi_1>=-\frac{1}{2}[|00>+|01>\ +|10>+|11>] \qquad (45)$$

where the state generated in the recalling procedure is neither maximally entangled nor the element of an orthonormal set of states. Thus the generated state in the recall mechanism is the fictitious and corrupt state and *hence the first state $|\psi_1>$ of Singh-Rajput MES cannot be a choice as valid memory state in the recall procedure with the given query.* Choosing the second and third states $|\psi_2>$ and $|\psi_3>$ of our new MES as the memory states in the recall mechanism with the given query, we have

$$F^{-1}I_{\bar{0}}\ FI_{M}F^{-1}I_{\bar{0}}\ FI_{00}|\psi_2>=-\frac{1}{2}[-|00>-|01>\ +|10>+|11>]$$

and

$$F^{-1}I_{\bar{0}}\ FI_{M}F^{-1}I_{\bar{0}}\ FI_{00}|\psi_3>=-\frac{1}{2}[-|00>-|01>\ +|10>+|11>] \qquad (46)$$

where the same state generated in both recall equations is corrupt state which is neither maximally entangled nor the element of an orthonormal set of states. *Thus none of these states $|\psi_2>$ and $|\psi_3>$ of Singh-Rajput MES can be a suitable choice as a valid memory state in the recall procedure with the given memory.* On the other hand when we choose the fourth state $|\psi_4>$ of this set of MES as the memory state then we get

$$F^{-1}I_{\bar{0}}\ FI_{M}F^{-1}I_{\bar{0}}\ FI_{00}|\psi_4>=-\frac{1}{2}[-|00>+|01>\ +|10>+|11>]=|\psi_1> \qquad (47)$$

where the recall procedure transforms the memory state $|\psi_4>$ in to the first state $|\psi_1>$ with the nature of maximal entanglement and orthonormal property left intact without affecting the chance of observation of the pattern represented in the given query. *Thus the fourth state $|\psi_4>$ of Singh-Rajput MES is most suitable as a valid memory state in the evolutionary recall procedure with the given query $|00>$.*

b) **Non-Evolutionary Approach of Pattern Recall**

Since the pattern recall mechanism in the QuAM requires the decoherence and collapse of the wave-function of the system, it can be argued that the pattern recall may be a non-unitary process. With this motivation we have constructed a new set of non-unitary operators for the pattern recall phase of QuAM as $\hat{R}$ and represented them as matrices $r_{\varphi\chi}$ indexed by column and row as the basis states φ and χ of the system

$$r_{\varphi\chi}=\begin{bmatrix}1\ if\varphi=\chi\ and\ h(\varphi,q)\geq 1\\ -1\ if\ h(\varphi,q)>h(\chi,q)\geq 1\\ 0\ \ otherwise\end{bmatrix} \qquad (48)$$

where $h(\varphi,q)$ etc are hamming functions and the character '?' matches any thing.

In two-qubit system to query the system with q= ‘11’ requires non-unitary operator;

$$\hat{R}^{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have applied this recall operator on Singh-Rajput MES, given by eqns. (5), chosen one by one as memory states. Choosing first of Singh –Rajput MESas memory state, we get

$$\hat{R}^{11}|\psi_1> = \frac{1}{2}\hat{R}^{11}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\,[|11>] \qquad (49)$$

which yields the requisite query without any spurious or corrupted state.*Thus the state* $|\psi_1>$ *of Singh-Rajput MES is the most suitable choice as the valid memory state in the non-evolutionary recall process with the given queryq= ‘11’.*

Choosing other states of Singh –Rajput MES as memory states, we get

$$\hat{R}^{11}|\psi_2> = [-|01> + \frac{1}{2}|11>]$$

$$\hat{R}^{11}|\psi_3> = [-|10> + \frac{1}{2}|11>] ;$$

$$\hat{R}^{11}|\psi_4> = [|01> + |10> - \frac{1}{2}|11>] ; \qquad (50)$$

Each of these equations gives spurious and undesired patterns accompanying the required query and *hence none of the last three states* $|\psi_2>, |\psi_3>$ *and* $|\psi_4>$ *of Singh-Rajput MES is suitable choice as memory state in this non-evolutionary recall process with the given query.*

On applying the recall operator $\hat{R}^{11}$ on the Bell’s MES, given by eqns. (4), chosen one by one as memory state with the given query ‘11’, we have

$$\hat{R}^{11}|\phi_1> = -\frac{i}{\sqrt{2}}[|01> + |10> - |11>];$$

$$\hat{R}^{11}|\phi_2> = \frac{1}{\sqrt{2}}[-|10> + |11>];$$

$$\hat{R}^{11}|\phi_3> = -\frac{i}{\sqrt{2}}[|01> + |10>];$$

$$\hat{R}^{11}|\phi_4> = \frac{1}{\sqrt{2}}[|01> - |10>]; \qquad (51)$$

where last two recall equations do not yield the required query pattern while first two equations yield the query pattern along with the unwanted spurious patterns. *Thus none of the Bell’s MES is suitable choice as valid memory state in this non-evolutionary recall process with the requisite query*.

We have also carried out the non-evolutionary recall process with the required query as the point ‘00’ or the pattern $|00>$. The recall operator for this query has been obtained as follows by using eqn. (48);

$$\hat{R}^{00} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Choosing first three states $|\psi_1>, |\psi_2>$ and $|\psi_3>$ of Singh-Rajput MES as memory states in this recall process with the given query, we get

$$\hat{R}^{00}|\psi_1> = \frac{1}{2}[-|00> + 2|01> + 2|10>],$$

$\hat{R}^{00}|\psi_2> = \frac{1}{2}[|00> - 2|01>],$

$\hat{R}^{00}|\psi_3> = \frac{1}{2}[|00> - 2|10>]$ (52)

None of these recall equations yield the requisite pattern, represented by the given query, free from the unwanted spurious patterns and *hence none of the first three states $|\psi_1>, |\psi_2>$ and $|\psi_3>$ of Singh-Rajput MES is suitable as valid memory state in this non-evolutionary recall process with the given query.*

On the other hand when this recall operator is applied on the fourth state $|\psi_4>$of Singh-Rajput MES, we get

$\hat{R}^{00}|\psi_4> = \frac{1}{2}[|00>]$ (53)

which gives the correct pattern represented by the given query '00' without any spurious pattern. *Thus the state $|\psi_4>$ of Singh-Rajput MES is the most suitable choice as the valid memory state in this non-evolutionary recall process with the given query '00'.*

Applying this recall operator $\hat{R}^{00}$ on Bell's MES one by one, we have

$$\hat{R}^{00}|\phi_1> = -\frac{i}{\sqrt{2}}[|00> - |01> - |10>],$$

$$\hat{R}^{00}|\phi_2> = \frac{1}{\sqrt{2}}[|00> - |01> - |10>],$$

$\hat{R}^{00}|\phi_3> = -\frac{i}{\sqrt{2}}[|01> + |10>],$

$\hat{R}^{00}|\phi_4> = \frac{1}{\sqrt{2}}[|01> - |10>]$ (54)

where last two recall equations do not yield the pattern required by the given query and the first two recall equations yield this pattern along with undesired spurious patterns. *Thus none of the Bell's MES is suitable as memory state in this recall procedure with the given query '00' also.*

The recall operator for the query '?1' with ? either 0 or 1 has been obtained as follows by using eqn. (48);

$$\hat{R}^{?1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (55)$$

Applying this recalling operator on the states of Singh- Rajput MES one by one, we get

$\hat{R}^{?1}|\psi_1>=\frac{1}{2}[\ |01> \ +|11>]$,

$\hat{R}^{?1}|\psi_2>=\frac{1}{2}[-\ |01> \ +|11>]$,

$\hat{R}^{?1}|\psi_3>=\frac{1}{2}[|01> +|11>]$

and $\qquad \hat{R}^{?1}|\psi_4>=\frac{1}{2}[|01> -|11>] \qquad (56)$

Each of these recall equations yields the patterns $|01>$and$|11>$, represented by the given query, with equal probability. *Thus all these states of Singh-Rajput MES are the most suitable choice as the valid memory in this non-evolutionary recall process also.*

On the other hand, choosing the Bell's MES one by one as the memory states in this recalling process with the given query, we get

$$\hat{R}^{?1}|\phi_1>=\frac{i}{\sqrt{2}}|11>; \quad \hat{R}^{?1}|\phi_2>=\frac{1}{\sqrt{2}}|11>;$$

$\hat{R}^{?1}|\phi_3>= -\frac{i}{\sqrt{2}}|11>; \quad \hat{R}^{?1}|\phi_4>= \frac{1}{\sqrt{2}}|01> \qquad (57)$

None of these equations yield the complete required pattern consisting of $|01>$ and $|11>$ represented by the given query. *Thus none of the Bell's MES is the good choice as the memory state in this recalling with this given query q='?1' also.*

Similarly for the query '0?' eqn. (48) leads to the following recalling operator;

$$\hat{R}^{0?}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

which gives

$\hat{R}^{0?}|\psi_1>=\frac{1}{2}[\ -|00> \ +|01>]$;

$\hat{R}^{0?}|\psi_2>=\frac{1}{2}[|00> \ -|01>]$;

$\hat{R}^{0?}|\psi_3>=\frac{1}{2}[\ |00> \ +|01>]$;

$\hat{R}^{0?}|\psi_4>=\frac{1}{2}[\ |00> \ +|01>]$; (58)

showing that *all Singh-Rajput MES are suitable for the choice of valid memory states in this non-evolutionary recall operation with the given query.* It has also been shown that *none of the Bell's MES is very suitable as the choice of valid memory states in this recalling operation.*

The similar results follow for the query '?0' also where the recalling operator is obtained as

$$\hat{R}^{?0}=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

which leads to the following sets of recall equations for Singh-Rajput MES and Bell' MES respectively;

$\hat{R}^{?0}|\psi_1>=\frac{1}{2}[-|00> +|10>];$

$\hat{R}^{?0}|\psi_2>=\frac{1}{2}[|00> +|10>];$

$\hat{R}^{?0}|\psi_3>=\frac{1}{2}[|00> -|10>];$

$\hat{R}^{?0}|\psi_4>=\frac{1}{2}[|00> +|10>];$ (59)

and

$$\hat{R}^{?0}|\phi_1>= -\frac{i}{\sqrt{2}}|00>; \quad \hat{R}^{?0}|\phi_2>= \frac{1}{\sqrt{2}}|00>;$$

$\hat{R}^{?0}|\phi_3>= -\frac{i}{\sqrt{2}}|10>; \quad \hat{R}^{?0}|\phi_4>= \frac{1}{\sqrt{2}}|10>$ (60)

where the first set of recall equations yields the full pattern represented by the given query and the second set in each case gives only the partial pattern. *Thus in this non-evolutionary recall process also the choice of Singh-Rajput MES as valid memory states is most suitable with the given query while the Bell's MES are not very suitable choice as the valid memory.*

From the foregoing analysis of the work carried out on the evolutionary and non-evolutionary recall operations it follows that these recall operations of quantum associative memory (QuAM) have been conducted separately through evolutionary as well as non-evolutionary processes in terms of unitary and non- unitary operators respectively by separately choosing our recently derived maximally entangled states (Singh-Rajput MES) and Bell's MES as memory states for various queries and it has been shown that in each case the choices of Singh-Rajput MES as valid memory states are much more suitable than those of Bell's MES. It has been demonstrated that in both the types of recall processes (evolutionary as well as non-evolutionary) the first and the fourth states of Singh-Rajput MES are most suitable choices as memory states for the queries '11' and '00' respectively while none of the Bell's MES is a suitable choice as valid memory state in these recall processes. It has been demonstrated that all the four states of Singh-Rajput MES are suitable choice as valid memory states for the queries '1?', '?1', '?0' and '0?' while none of the Bell's MES is suitable choice as the valid memory state for these queries also.

All these results on the superiority and suitability of our new MEs (Singh-Rajput MES), given by eqns. (5), as the search states (data base) for correct pattern classification through both the standard processes (Grover's algorithm and Ventura' algorithm), pattern memorisation through evolutionary process, and pattern recall through evolutionary as well as non-evolutionary processes confirm the following applicability of these new MES:

i) First MES $|\psi_1>$ of our new Eigen- Basis (Singh-Rajput Eigen-basis) is the best choice as the search state for total desired classification of the patterns $|0?>$, correct classification of the pattern $|00>$, memorisation (storage) of the pattern $|11>$, and recall of the pattern $|11>$ through both, evolutionary as well as non-evolutionary approaches.

ii) Second MES $|\psi_2>$ of our new Eigen –Basis is the best choice as the search state for the total desired classification of the patterns $|0?>$, correct classification of the pattern $|01>$, and memorisation of the pattern $|11>$

iii) Third MES $|\psi_3>$ of our new Eigen-Basis is the best choice as the search state for the total desired classification of the patterns $|1?>$, correct classification of the pattern $|10>$, and memorisation of the pattern $|10>$.

iv) Fourth MES $|\psi_4>$of our new Eigen-Basis is the best choice as the search state for the total desired classification of the patterns $|1?>$, correct classification of the pattern $|11>$, memorisation of the pattern $|10>$. and recall of the pattern $|00>$ through both, evolutionary as well as non-evolutionary approaches.

v) All the four MES of our new Eigen-Basis are suitable for recalling the patterns $|1?>, |0?>$, $|?0>$, and $|?1>$, where the symbol '?' denotes 0 or 1, through both evolutionary as well as non-evolutionary approaches.

D) QUANTUM ENCODING AND ENTANGLEMENT IN TERMS OF PHASE OPERATORS ASSOCIATED WITH HARMONIC OSCILLATOR.

Keeping in view that a major obstacle to universal quantum computing [30] and the quantum information processing [31] is the limit on number of coupled qubits that can be achieved in a physical system [32]and use of d-dimensional system or qudit in quantum computing enables a much more compact and efficient information encoding than for qubit computing, realization of qudit quantum computation has been presented [15] in terms of number operator and phase operators associated with one-dimensional harmonic oscillator and it has been demonstrated that the representations of generalized Pauli group, viewed in harmonic oscillator operators, allows the qudits to be explicitly encoded in such systems.The non-Hermitian quantum phase operators contained in decomposition of the annihilation and creation operators associated with harmonic oscillator have been analysed in terms of semi unitary transformations (SUT)and it has been shown that the non-vanishing analytic index for harmonic oscillator leads to an alternative class of quantum anomalies. Choosing unitary transformation and the Hermitian phase operator free from quantum anomalies, the truncated annihilation and creation operators have been obtained for harmonic oscillator and it has been demonstrated that any attempt of removal of quantum anomalies leads to absence of minimum uncertainty.It has also been demonstrated that despite the issues involving d$\rightarrow \infty$ phase operators, universal qudit quantum computation is well defined for the finite d and a criterion for existence of topological phases, satisfied by a wide class of states, follows immediately from the theory of polynomial invariants.

To undertake the study of qudits in energy eigen space of harmonic oscillator, aqudit is realised as a state in a d-dimensional Hilbert space H_d, with a computational basis $\{ \left| s > : s = 0,1, \ldots \ldots d-1 \right\}$.

A basis for unitary operator on H_d is given by the generalized Pauli operators [33]:

$(X_d)^a (Z_d)^b, \; a, b \in 0,1, \ldots \ldots . d-1$ (61)

where $X_d \left| s > = \right| s+1 > (mod\ d)$

and $Z_d \left| s > = \exp(\frac{2\pi i s}{d}) \right| s >$ (62)

generate the non-commutative generalized Pauli group. For qudits in the Eigen space of harmonic oscillator, weobtain a generalized Pauli group generated by the number operator $\widehat{N}$ and the phase operator $\hat{\theta}$. Here the Pauli operators X_d and $Z_d$have been realized as operators that act naturally on the space H_d of dimension d spanned by the states of harmonic oscillator by using the computational basis to be the set of harmonic oscillator energy Eigen states of no more than $(d-1)$ bosons:

$\left| s > = \right| n = s >, s = 0,1, \ldots \ldots ., d-1$ (63)

where $\widehat{N} \left| n > = n \right| n >$

The generalized Pauli group (the operators on this subspace of the harmonic oscillator) is realized as

$$X_d \rightarrow \sum_{n=0}^{d-1} \left| s+1 > < s \right|,$$

$$Z_d \rightarrow \exp(\frac{2\pi i N}{d}) \quad (64)$$

which are unitary on H_d . The operator X_d may also be written as

$X_d = \exp(2\pi i \hat{\theta}_z / d)$ (65)

where the Hermitian operator $\hat{\theta}_z$ is the Pegg-Bernett phase operator [34]. The realization given by eqns. (64) and (65) enables us to achieve the $d \rightarrow \infty$ limit which yields continuous-variable quantum computation where the computational basis remains the harmonic oscillator energy eigen states followed by eqn. (63). It provides the necessary theoretical tools for developing the qudit quantum computation in terms of phase operators associated to harmonic oscillator, constructed in the following sub-sections.

i) Phase Operators Associated with Harmonic Oscillator

The quantum phase operator contained in polar decomposition of the annihilation and creation operators, and also the number operator, associated with harmonic oscillator has been exhaustively discussed and it has been shown that the Hermitian phase operator leading to vanishing index cannot be consistently defined. We have examined this problem from the first principle using ladder method for one dimensional oscillator with its annihilation and creation operators, respectively, given by

$$\hat{a} = |0><1| + \sqrt{2}|1><2| + \sqrt{3}|2><3| + \ldots\ldots..\sqrt{n}|n-1><n| + \cdots \quad (66)$$

$$\hat{a}^{\dagger} = |1><0| + \sqrt{2}|2><1| + \sqrt{3}|3><2| + \ldots\ldots..\sqrt{n}|n><n-1| + \cdots \quad (67)$$

which give number operator as

$$\hat{N} = \hat{a}^{\dagger}\hat{a} = |1><1| + 2|2><2| + 3|3><3| + \cdots n|n><n| + \cdots \quad (68)$$

From relations (66) and (67) we also have

$$\hat{a}\hat{a}^{\dagger} = |0><0| + 2|1><1| + 3|2><2| + |\ldots n|n-1><n-1| \quad (69)$$

From these equations it is obvious that only one basis vector (i.e., $|0>$) satisfy the condition

$$\hat{a}^{\dagger}\hat{a}|n> = 0 \quad (70)$$

and no basis vector satisfies the condition

$$\hat{a}\hat{a}^{\dagger}|n> = 0 \quad (71)$$

Thus if we designate $(\dim\ker \hat{a}^{\dagger}\hat{a})$ as the number of normalizable basis vectors $|n>$ satisfying the condition (70) and (dim ker$\hat{a}\hat{a}^{\dagger}$)as the number of basis vectors satisfying condition (71), then we have the index relation

$$\dim\ker \hat{a}^{\dagger}\hat{a} - dim\ ker\ \hat{a}\hat{a}^{\dagger} = 1, \quad (72)$$

for the harmonic oscillator. This index remains invariant under continuous deformations and one cannot relate the representation spaces of annihilation operators with different indices by a unitary transformation.

Dirac first proposed [35] the polar decomposition of the annihilation operator as

$$\hat{a} = e^{i\emptyset}\sqrt{\hat{N}} = \hat{U}\,[\hat{a}^{\dagger}\hat{a}]^{1/2} \quad (73)$$

where $\hat{U} = e^{i\emptyset}$ and $\emptyset$ is the phase operator. This relation gives

$$\hat{a}^{\dagger} = (\hat{N})^{1/2}\hat{U}^{\dagger} \quad (74)$$

and hence $\hat{N} = (\hat{N})^{1/2}U^{\dagger}\hat{U}(\hat{N})^{1/2}$

which is possible only when

$$\hat{U}^{\dagger}\hat{U} = \hat{I} \quad (75)$$

where$\hat{I}$ is the identity operator. But the relations (73) and (74) also give

$$\hat{a}\hat{a}^{\dagger} = \hat{U}(\hat{N})^{\frac{1}{2}}(\hat{N})^{\frac{1}{2}}\hat{U}^{\dagger} = \hat{U}\hat{N}\hat{U}^{\dagger} = \hat{U}(\hat{a}^{\dagger}\hat{a})\hat{U}^{\dagger} \quad (76)$$

This relation denies the unitary property of operator $\hat{U}$ in view of index relation (72) since the operators $\hat{a}\hat{a}^{\dagger}$ and $\hat{a}^{\dagger}\hat{a}$ cannot be connected by unitary operator for the index relation (72) to hold good. Thus

$$\hat{U}\hat{U}^{\dagger} \neq I \quad (77)$$

Operator satisfying conditions (75) and (77) is semi-unitary and not unitary and hence the phase operator introduced in eqn. (73) is not Hermitian:

$$\emptyset^{\dagger} \neq \emptyset \quad (78)$$

It shows the absence of the Hermitian phase operator for harmonic oscillator in the framework of index theory. To meet this requirement the phase operator suggested by Susskind and Glower [36] has been written as

$$\hat{U} = e^{i\emptyset} = \left|0 >< 1\right| + \left|1 >< 2\right| + \left|2 >< 3\right| + \cdots$$

$$= \sum_{n=0}^{\infty} \left|n >< n+1\right| \quad (79)$$

which gives

$$\hat{U}\hat{U}^{\dagger} = \left|0 >< 0\right| + \left|1 >< 1\right| + \left|2 >< 2\right| + \cdots$$

$$= \sum_{n=0}^{\infty} \left|n >< n\right| = I \quad (80)$$

and $$\hat{U}^{\dagger}\hat{U} = \left|1 >< 1\right| + \left|2 >< 2\right| + \cdots$$

$$= I - \left|0 >< 0\right| = \hat{P} \quad (81)$$

showing that the operator $\hat{U}$ is semi-unitary and not unitary operator. Accordingly, the transformation (73) is semi-unitary transformation. Eqn. (79) shows that the operator $\hat{U} = e^{i\emptyset}$ carries a unit index while $\hat{U}^{\dagger}$ is empty and hence

$$\dim \ker \hat{U}^{\dagger}\hat{U} - \dim \ker \hat{U}\,\hat{U}^{\dagger} = 1 \quad (82$$

which is equivalent to index relation (72).

Equations (80) and (81) show that any transformation involving operator $\hat{U}$will be a semi-unitary transformation (SUT). The operator $\hat{P}$ introduced in equation (81) satisfies the following conditions;

$$\hat{P}^{\dagger} = \hat{P} \quad \text{and} \quad \hat{P}^2 = \hat{P} \quad (83)$$

showing that $\hat{P}$ is projection operator with eigen values 0 and 1. It may readily be shown that the projection operator $\hat{P}$ introduced in eqn. (81) commutes with the Hamiltonian of harmonic oscillator and hence it has Eigen vectors common with harmonic oscillator. Thus we have

$$\hat{P}\left|0 > = \right|0 >$$

and $$\hat{P}\left|n+1 > = \right|n+1 > \quad (84)$$

showing that under the projection operator $\hat{P}$ the full Hilbert space $\mathcal{H}$of harmonic oscillator is projected into two subspaces:

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ (85)

where the subspace $\mathcal{H}_0$ is constituted by the state $|0>$ and the subspace $\mathcal{H}_1$ is constituted by the states $|n+1>$.

To find the general structure of the operator $\hat{P}$ in the above mentioned basis, we started with the complete set |n>of harmonic oscillator (for n=0.1,2,…) and constructed

$|n>_+ = \hat{U}^\dagger | n>$(86)

*w*hich gives

$\sum_{n=0}^{\infty} |n>_{++}<n| = \hat{U}^\dagger \sum_{n=0}^{\infty} | n><n | \hat{U} = \hat{U}^\dagger \hat{U} = \hat{P}$ (87)

showing that the states $|n>_+$ do not form the complete set.

We obviously have

$\hat{P}|n>_+ = |n>_+$ (88)

and hence we may write

$|n>_+ = |p=1>$ (89)

where p denotes the Eigen values of $\hat{P}$. The complete set of the Eigen states is formed by the states $|0> = |p=0>$and $|n>_+$ given by eqn. (89). Thus we have

$$|0><0| + \sum_n |n>_{++}<n| = \hat{I}$$

or $\sum_n |n>_{++}<n| = \hat{I} - |0><0|$ (90)

which is the same result as eqn. (87). Thus the matrix of the operator $\hat{P}$ is diagonal with the general structure

$$\hat{P} = \begin{bmatrix} 0 & 0\ldots & 0 & 0 \\ 0 & 1\ldots & 0 & 0 \\ 0 & 0\ldots & \vdots & \vdots \\ 0 & 0\ldots & 0 & 1 \end{bmatrix} = I - |0><0| \quad (91)$$

where I is unit matrix.

From the SUT transformation, given by eqn. (86), we get

$._+<n|n>_+ = <n|\hat{U}\hat{U}^\dagger|n> = <n|n>$ (92)

showing that the orthonormality of states $|n>$ implies the orthonormality of states $|n>_+$. Equations (90) and (92) demonstrate that under the SUT given in eqn. (86), the normalization of the states of harmonic oscillator is maintained while the completeness is violated.

We have also considered the semi-unitary transformation:

$|n>_- = \hat{U}|n>$ (93)

to have

$\sum_{n=0} |n>_{--}<n| = \sum_{n=0} \hat{U}|n><n|\hat{U}^\dagger = \hat{U}\hat{U}^\dagger = \hat{I}$ (94)

showing that completeness of the states $|n>$ implies the completeness of the states $|n>_-$. But the SUT (93) also give

$$._- < m|n>_- = < m|\hat{U}^\dagger \hat{U}|n> = < m|\hat{P}|n> \neq \delta_{mn} \quad (95)$$

Thus the SUT given by eqn. (93) maintains completeness relation but fails to maintain normalisation condition. Furthermore, under this SUT we have

$$|0>_- = \hat{U}|0> = 0 \quad (96)$$

showing that this SUT destroys the ground state $|0>$.

The semi-unitary operator $\hat{U} = e^{i\emptyset}$ with non-Hermitian phase operator $\emptyset$, introduced in eqn. (73), satisfies the following commutation relation with number operator $\hat{N}$ given by eqn. (68);

$$[\hat{N}, \hat{U}] = -\hat{U}^\dagger \quad (97)$$

which is compatible with the commutation relation

$$[\hat{N}, i] = i \quad (98)$$

Then the functions analogous of cosine and sine, defined as

$$C(\emptyset) = \frac{1}{2}[\hat{U} + \hat{U}^\dagger]$$

and $$S(\emptyset) = \frac{1}{2i}[\hat{U} - \hat{U}^\dagger] \quad (99)$$

give the anomalous commutator

$$[\ C(\emptyset), S(\emptyset)] = -\frac{1}{2i}\ |0><0| \quad (100)$$

and the anomalous identity

$$[C(\emptyset)]^2 + [S(\emptyset)]^2 = 1 - |0><0| \quad (101)$$

These functions also satisfy the following anomalous commutation relations with number operator:

$$[\hat{N}, C(\emptyset)] = -i\ S(\emptyset)$$

and $$[\hat{N}, S(\emptyset)] = iC(\emptyset) \quad (102)$$

Thus we may regard the phase operator problem associated with non-vanishing analytic index for harmonic oscillator as alternative class of quantum anomalies. Index relation (72), responsible for these anomalies, is satisfied in a manner analogous to chiral anomaly in quantum field theory [37]. This connection between the non-zero index and the chiral anomaly appears in the transparent manner in the Euclidean path integral formulation of anomalies [38].

ii) Phase Operator Free From Quantum Anomalies

A possible choice of Hermitian phase operator, free from quantum anomalies, has been obtained from theinfinite dimensionaloperator

$$e^{i\Psi} = e^{i\theta} + \left|s+1 >< s+2\right| + \cdots + \left|2s >< 2s+1\right|$$

$$+e^{i\theta_1}\left|2s+1 >< s+1\right| + \left|2s+2 >< 2s+3\right| + \cdots + \left|3s+1 >< 3s+2\right|$$

$$+e^{i\theta_2}\left|3s+2 >< 2s+2\right| + \cdots \qquad (103)$$

where $\theta_1, \theta_2, \ \ldots$ are real constants and θ is the Hermitian phase operator (i.e., $\theta = \theta^{\dagger}$) defined in the truncated (s+1) dimensional space as

$$e^{i\theta} = \left|0 >< 1\right| + \left|1 >< 2\right| + \cdots + \left|s-1 >< s\right| + e^{i(s+1)\theta_0}\left|s >< 0\right| \qquad (104)$$

where θ_0 is an arbitrary c-number constant. The phase operator $e^{i\theta}$ is obviously unitary in $(s+1)$ dimensions and the$(s+1)$ dimensional truncated creation and annihilation operators are defined as

$$\hat{a}_s = e^{i\theta}(\widehat{N})^{\frac{1}{2}} = \left|0 >< 1\right| + \sqrt{2}\left|1 >< 2\right| + \sqrt{3}\left|2 >< 3\right| + \cdots + \sqrt{s}\left|s-1 >< s\right|$$

and

$$\hat{a}_s{}^{\dagger} = e^{-i\theta}(\widehat{N})^{\frac{1}{2}} = 1 >< 0\left| + \sqrt{2}\right|2 >< 1\left| + \sqrt{3}\right|3 >< 2\left| + \cdots + \sqrt{s}\right|s >< s-1\left| \right. \qquad (105)$$

which give

$$[\hat{a}_s, \hat{a}_s{}^{\dagger}] = \hat{I} - (s+1)\left|s >< s\right|$$

and $\qquad \hat{a}_s\hat{a}_s{}^{\dagger}\left|s >= 0\right.$

Hence, with the phase operator of eqn. (104), we have

$$\dim\ker \hat{a}_s{}^{\dagger}\,\hat{a}_s - \dim\ker \hat{a}_s\hat{a}_s{}^{\dagger} = 0 \qquad (105a)$$

which is the index relation required for the Hermitian nature of phase operator. For this choice of phase operator relations (99) reduce to

$$C(\theta) = \cos\theta \quad \text{and } S(\theta) = \sin\theta \qquad (106)$$

and the anomalies in the commutation relation (100) and the identity (101) are removed. These characteristics of phase operator of eqn. (104) are retained in the phase operator Ψ defined by eqn. (103) and hence the operator$e^{i\Psi}$ is unitary and phase operator Ψ is Hermitian.

For any normalized state $\left|n >\right.$ of the number operator associated with harmonic oscillator, we have

$$< n\left|e^{i\Psi}\right|n > = 0$$

and the operator

$$\hat{A}_s = e^{i\Psi}(\widehat{N})^{\frac{1}{2}} \qquad (107)$$

satisfies the index relation

$$\dim\ker \hat{A}_s{}^{\dagger}\,\hat{A}_s - \dim\ker \hat{A}_s\hat{A}_s{}^{\dagger} = 0 \qquad (108)$$

which is equivalent to the index relation (105a).

In spite of the removal of quantum anomalies, natural modification in index relation and the Hermitian nature of phase operator Ψ, the operator $\hat{A}_s$, defined by eqn. (107), is not related with annihilation operator of eqn. (66) by unitary transformation for a finite value of s and the limit $s \to \infty$ becomes a singular point of equation (108). This fact leads to the absence of minimum uncertainty state for the operator $e^{i\Psi}$ in the characteristically quantum domain [37]. If one uses an analogy between the phase operator and the chiral anomaly, the index relation corresponds to the quantum anomaly which is clearly recognised only when one carefully analyses the dependence of the matrix elements of various operators of harmonic oscillator on the cut off parameter s [37]. Despite the issues involving d→ ∞ phase operators, universal qudit quantum computation is well defined for the finite d [39]. These requirements may be achieved in terms of optical realisation where harmonic oscillators are realised as modes in a cavity [40]. Quantum computation with multiple qudits may be performed by coupling several modes in a single cavity where each mode realizes a single qudit [40]. The representations of the generalized Pauli group, viewed in terms of number and phase operators for harmonic oscillator, allows for qudits to be explicitly encoded in terms of these factors.

8. Publications Based on the Work Carried out in the Project.

Based on the research work carried out in in this project, the following **08 (Eight)** research papers have been published in the journals of international repute (Springer and Elsevier) (Offprints attached)

1. **Pattern Classification Using Grover's and Ventura's Algorithms in a Two-qubits System**
   Manu Pratap Singh, Kishori Radhey & B. S. Rajput
   International Journal of Theoretical Physics, SPRINGER, 57 (3) (2018) 692 – 705, **Impact factor 1.184, ISSN 0020-7718**
2. **Multilayer feed forward neural networks for non-linear continuous bidirectional associative memory**
   Manu Pratap Singh & V K Saraswat
   Journal of Applied Soft Computing, Journal of ELSEVIER (Science Direct), 61 (2017), 700-713, **Impact factor 2.100**
3. **New Maximally Entangled States for Pattern Associations through Evolutionary Process in a Two-Qubit System.**
   Manu Pratap Singh & B. S. Rajput
   International Journal of Theoretical Physics, SPRINGER, 55 (12) (2017) 3269 – 3287, **Impact factor 1.184, ISSN 0020-7718**
4. **Classification of Pattern Representing Apples and Oranges in three-qubit system**
   Manu Pratap Singh, Kishori Radhey, V. K. Saraswat & Sandeep Kumar
   Quantum Processing System, SPRINGER, 16 (1) (2017), **Impact factor 1.840, ISSN** 1570-0755
5. **Simultaneous Classification of Oranges and Apples Using Grover's and Ventura' Algorithms in a Two-qubit System.**
   Manu Pratap Singh, Kishori Radhey & Sandeep Kumar
   International Journal of Theoretical Physics, SPRINGER, 56 (6) (2017) 2521-2534, **Impact factor 1.184, ISSN** 0020-7718
6. **Process of Quantum Associative Memory (QuAM) through New Maximally Entangled States (Singh – Rajput MES)**
   Manu Pratap Singh & B. S. Rajput

International Journal of Theoretical Physics, SPRINGER, 55 (7) (2016) 3207 – 3219, **Impact factor 1.184, ISSN 0020-7718**

7. **Quantum Encoding and Entanglement in Terms of Phase Operators Associated with Harmonic Oscillator**
   Manu Pratap Singh & B. S. Rajput
   International Journal of Theoretical Physics, SPRINGER, 55 (10) (2016) 4393 – 4405, **Impact factor 1.184, ISSN 0020-7718**
8. **Application of Singh-Rajput MES in Recall Operations of Quantum Associative Memory for a Two-Qubit System**
   Manu Pratap Singh & B. S. Rajput
   International Journal of Theoretical Physics, SPRINGER, 54 (10) (2015) 3443 – 3866, **Impact factor 1.184, ISSN 0020-7718**

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